

Coxeter group structure of cosmological billiards on compact spatial manifolds

Marc Henneaux* and Daniel Persson†

*Physique Théorique et Mathématique,
Université Libre de Bruxelles & International Solvay Institutes,
ULB-Campus Plaine C.P.231, B-1050 Bruxelles, Belgium
E-mail: henneaux, dpersson@ulb.ac.be*

Daniel H. Wesley

*Centre for Theoretical Cosmology,
Department of Applied Mathematics and Theoretical Physics, Cambridge University,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
E-mail: D.H.Wesley@damp.cam.ac.uk*

ABSTRACT: We present a systematic study of the cosmological billiard structures of Einstein- p -form systems in which all spatial directions are compactified on a manifold of nontrivial topology. This is achieved for all maximally oxidised theories associated with split real forms, for all possible compactifications as defined by the de Rham cohomology of the internal manifold. In each case, we study the Coxeter group that controls the dynamics for energy scales below the Planck scale as well as the relevant billiard region. We compare and contrast them with the Weyl group and fundamental domain that emerge from the general BKL analysis. For generic topologies we find a variety of possibilities: (i) The group may or may not be a simplex Coxeter group; (ii) The billiard region may or may not be a fundamental domain. When it is not a fundamental domain, it can be described as a sequence of pairwise adjacent chambers, known as a *gallery*, and the reflections in the billiard walls provide a non-standard presentation of the Coxeter group. We find that it is only when the Coxeter group is a simplex Coxeter group, and the billiard region is a fundamental domain, that there is a correspondence between billiard walls and simple roots of a Kac-Moody algebra, as in the general BKL analysis. For each compactification we also determine whether or not the resulting theory exhibits chaotic dynamics.

KEYWORDS: Discrete and Finite Symmetries, Spacetime Singularities.

*Also at *Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*

†Also at *Fundamental Physics, Chalmers University of Technology, SE-412 96, Göteborg, Sweden*

Contents

1. Introduction and motivation	2
1.1 Cosmological billiards and overextended U-duality	2
1.2 Intermediate asymptotics	3
1.3 Summary of results	4
1.4 Outline of the paper	5
2. Coxeter billiards and compactification	5
2.1 Cosmological billiards and Coxeter groups	5
2.1.1 The BKL-limit, billiards and Kac-Moody algebras	5
2.1.2 The “uncompactified billiard”	7
2.2 Compactification and cohomology	8
2.2.1 Selection rule	8
2.3 Gram matrices and Coxeter groups	9
2.3.1 Convex polyhedra	9
2.3.2 Relative positions of walls in hyperbolic space	10
2.3.3 Acute-angled polyhedra	10
2.3.4 Coxeter polyhedra and the billiard group	11
3. General results	12
3.1 Rules of the game	12
3.2 Describing the billiard group after compactification	13
3.2.1 The formal Coxeter group	13
3.2.2 The billiard group and its fundamental domain	14
3.2.3 Non-standard presentations of Coxeter groups	16
3.3 Fundamental domains, chamber complexes and galleries	16
3.3.1 Chambers and galleries	16
3.3.2 The billiard region as gallery	18
3.4 Determining the chaotic properties after compactification	21
4. Selected examples	23
4.1 The F_4 -sequence	23
4.1.1 The $b_1(\mathcal{M}_5) = 0$ compactification	24
4.1.2 The $b_1(\mathcal{M}_5) = b_2(\mathcal{M}_5) = 0$ compactification	24
4.2 The E_7 -sequence	25
4.2.1 Dominant wall structure after compactification	25
4.2.2 Non-standard presentations of E_7^{++}	27
4.2.3 The volume of the billiard gallery in the case $b_1(\mathcal{M}_8) = 0$	29
5. Concluding remarks	33

1. Introduction and motivation

Generic solutions to Einstein’s gravity coupled to dilatons and p -forms in the neighbourhood of a spacelike singularity possess a surprisingly rich and complicated structure. This was first revealed by Belinskii, Khalatnikov and Lifshitz (BKL), who showed that in vacuum four-dimensional Einstein gravity the dynamical approach to the singularity is oscillatory and chaotic [1–3] (see also [4]). Even though the chaotic oscillations disappear for pure gravity in ten dimensions or higher [5, 6], they are present in the bosonic sectors of all supergravities related to string and M-theory due to the presence of p -forms [7]. Chaos is revealed by casting the dynamics of gravity at each spatial point as billiard motion in a finite-volume region of an auxiliary hyperbolic space [8–10]. This type of dynamics has been extensively studied in mathematics and is well known to exhibit chaotic behaviour.

1.1 Cosmological billiards and overextended U-duality

The BKL analysis reveals a connection between gravitational dynamics and Kac-Moody algebras. For certain theories, including all maximal supergravities, the region where the billiard dynamics takes place can be identified with a bounded region within the Cartan subalgebra \mathfrak{h} of a Lorentzian Kac-Moody algebra \mathfrak{g} . In fact, the billiard motion is confined to the fundamental Weyl chamber in \mathfrak{h} and the geometric reflections of the billiard generate the Weyl group $\mathfrak{W}[\mathfrak{g}]$ of \mathfrak{g} . This connects the chaotic nature of certain supergravity theories to the hyperbolic nature of the underlying Kac-Moody algebra [11]. For example, the algebras whose Weyl groups control the dynamics of the string-related supergravities are E_{10} (SUGRA₁₁, Type IIA & IIB), BE_{10} (Heterotic, Type I) and DE_{10} (pure supergravity), which are all hyperbolic [12]. The Kac-Moody algebras governing the BKL behaviour are closely linked to the “U-duality” algebras appearing in compactifications of Einstein-dilaton- p -form systems to three dimensions: if \mathfrak{u} is the symmetry algebra in three dimensions, then the Kac-Moody algebra \mathfrak{g} controlling the dynamics in the BKL-limit is the “overextension” \mathfrak{u}^{++} of the U-duality algebra \mathfrak{u} [13] (see [14, 15] for details on U-duality).¹ By computing the billiard directly in three dimensions where the U-duality symmetry is manifest, and using the invariance of the billiard structure under toroidal compactification, it has been shown that taking the BKL-limit precisely mimics the overextension procedure described in [17].² In this way the BKL-limit “unveils” an algebraic structure which could play the role of a fundamental underlying symmetry of the

¹We assume the real Lie algebra \mathfrak{u} to be split. The BKL rules for the general case are given in [16].

²In [17] it is called the “canonical hyperbolic extension”.

theory [18]. This also lends indirect support for Julia’s conjecture that E_{10} is the symmetry of eleven-dimensional supergravity compactified on T^{10} [19, 20].³

1.2 Intermediate asymptotics

The original BKL analysis is classical and has been pushed all the way to the singularity [1, 2]. As such it is valid for any spatial topology. In the approach to a spacelike singularity there is an asymptotic decoupling of spatial points, and the dynamics becomes “ultralocal”. These results (decoupling of spatial points and chaotic oscillations) are by now well supported by extensive analytical and numerical evidence; a non-exhaustive list of references include [34–42].

The classical analysis has however obvious limitations and it is not clear what becomes of the BKL results for energy scales above the Planck scale, where quantum gravity effects cannot be ignored. In the standard BKL analysis, which ignores quantum effects, no walls are removed as the big crunch is approached. But, when some spatial dimensions are compact, there is a wide range of initial conditions for which walls corresponding to massive modes are always subdominant until the universe enters the quantum regime, and these walls are not relevant for the billiard analysis while the universe is described by classical physics [43, 44]. For this broad set of initial conditions, there is no epoch in which the usual classical BKL analysis, with the full set of walls, applies.

For this reason, it is of interest to consider the regime of *intermediate asymptotics* where the curvature is much smaller than the Planck curvature but where the billiard analysis applies (see [45]). In that pre-Planck regime, it is not true that the topology of spacetime is irrelevant [43]. A modification might arise in the presence of p -forms because the massless spectrum of p -forms in the lower-dimensional theory depends on the de Rham cohomology of the internal manifold, and since massless degrees of freedom dominate in the BKL-limit before reaching the Planck scale [43], non-trivial compactification eliminates billiard walls corresponding to degrees of freedom which are rendered massive in the compactification. Depending on which walls are removed by the compactification, a chaotic theory can be rendered non-chaotic. Note that the suppression of massive modes exhibited in [43] is a classical result, which follows from the virial theorem. However, one could argue that it is in fact also true quantum mechanically (in the intermediate regime where the geometry can be treated as classical but the matter fields are quantum-mechanical), for standard methods reveal that the expected energy density ρ_{pp} of pair-produced particles is at most [46]

$$\rho_{pp} \sim H^2 e^{-m/H}, \quad (1.1)$$

³In this paper we shall mostly focus on the “overextended” Kac-Moody algebras, which are closely linked to compactifications to one (timelike) dimension. For related work on affine Kac-Moody algebras, i.e., extended Lie algebras, in the context of reductions to two dimensions, see, e.g., [21–27]. So called “very extended” Kac-Moody algebras have also been extensively studied in the literature. Most notably, the Lorentzian Kac-Moody algebra E_{11} has been put forward as a possible underlying symmetry of eleven-dimensional supergravity (and, perhaps, M-theory) [19, 28–31]. See also the recent work [32, 33] in favour of this conjecture.

where H is the effective Hubble parameter and m the particle mass. The exponential suppression of massive mode pair production follows the point particle intuition [47], with an even greater suppression expected for string pair production [48].

There are a number of reasons to better understand the interplay between BKL dynamics and compactification. On the physics side, the BKL-limit with compact internal spaces is relevant for certain types of cyclic or “pre-big bang” cosmological models built from string or M-theory (see, e.g., [49–52] and references therein). Cosmological models with a big crunch/big bang transition rely on a smooth collapsing phase as they approach the big crunch singularity, hence chaotic BKL oscillations close to the singularity are a potential problem for these models. If chaos can be removed by interpreting our four-dimensional world as an effective description of a more fundamental higher-dimensional theory where the “troublesome” billiard walls are eliminated through the cohomology of the internal space, the problems with BKL chaos in these models may be circumvented. On the mathematical side, the billiard regions and the reflection groups that emerge after compactification possess a rich structure which deserves investigation.

1.3 Summary of results

The compactification analysis was recently carried out for eleven-dimensional supergravity and heterotic supergravity in [44]. This showed that, in most cases, the BKL-limit does not produce a dominant wall set with the properties required of a valid Kac-Moody root system. In this paper, we extend the investigation initiated in [44] by studying compactifications of all the maximally oxidised theories originally classified by Breitenlohner, Maison and Gibbons [53] and further analysed in [54, 55]. These theories associate an Einstein- p -form system with any algebra \mathfrak{u} in the $A - G$ Cartan classification. The Einstein- p -form system, when compactified on a torus, yields a scalar coset model in $D = 3$ spacetime dimensions with U-duality symmetry algebra given by a split real form of \mathfrak{u} , and whose BKL billiard is associated with the overextension \mathfrak{u}^{++} of this split real form. We perform an exhaustive analysis of all compactifications (defined by their vanishing Betti numbers) of all of the $A - G$ Einstein- p -form systems. We present the Dynkin diagrams or Coxeter graphs describing the resulting billiards, and determine whether chaos is present in all possible cases.

We also elucidate the structure of the group of reflections in the dominant billiard walls that survive compactification, and its relation with the billiard region. The group is always a Lorentzian Coxeter group but the billiard region is not necessarily a fundamental domain of this Coxeter group. Rather it may correspond to a union of images of the fundamental Weyl chamber of the hyperbolic algebra \mathfrak{u}^{++} associated with the uncompactified theory. Using techniques from the theory of buildings we show that this region has a mathematical description as a sequence of pairwise adjacent chambers, known as a *gallery*, inside the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{u}^{++}$. Moreover, the resulting Coxeter group is then described by a non-standard presentation, with additional relations between the fundamental reflections, apart from the standard Coxeter relations. The Coxeter group itself might not be a simplex Coxeter group (i.e., a Coxeter group with fundamental region that is a simplex). It is only when the Coxeter group is a simplex Coxeter group, and the billiard region is a

fundamental domain, that there is a correspondence between billiard walls and simple roots of a Kac-Moody algebra. This situation is generically absent in the compact setting (in the intermediate regime considered here).

1.4 Outline of the paper

Our paper is organized as follows. Section 2 gives a quick review of crucial elements of the billiard construction, with emphasis on the associated Coxeter group structure and its underlying geometric properties. In particular, we analyze some properties of convex polyhedra in hyperbolic space and explain how they can be used to understand certain features of Coxeter groups. In section 3 we give some useful facts regarding the wall systems after compactification. We extend the set of theorems introduced and proven in [13] regarding the set of dominant walls for a given Einstein-dilaton- p -form system, to include the cases where individual p -form components are eliminated. We introduce the notions of *formal Coxeter group* and *billiard group* that will play important roles in subsequent developments. These concepts enable us to discuss general features of the Coxeter group structure of the reduced billiard, and we analyze in detail the properties of the new billiard domain. Most importantly, we show that this domain corresponds to a gallery, and we demonstrate how the structure of this gallery is related to the presentation of the associated Coxeter group. In section 4 we describe in detail several especially illuminating cases in order to illustrate the techniques that we use, and to put the abstract results of section 3 on a concrete footing. In section 5 we end with concluding remarks and suggestions for future research. Finally, a complete classification of all possible compactifications of maximally oxidised theories is given in the appendix.

2. Coxeter billiards and compactification

In this section we set the stage for the analysis carried out in sections 3 and 4. We give a brief review of the billiard interpretation of the dynamics in the BKL-limit, without attempting completeness; more details can be found in a number of review articles [9–12]. Following [43, 44] we also describe how compactification on manifolds of non-trivial topology modifies the billiard dynamics by projecting out walls associated with p -form fields which are rendered massive in the compactification. Furthermore, this section introduces some useful technology in the theory of Coxeter groups that will be crucial for understanding the main results of section 3. In particular, we analyze certain properties of convex polyhedra in hyperbolic space, and the associated Gram matrices.

2.1 Cosmological billiards and Coxeter groups

2.1.1 The BKL-limit, billiards and Kac-Moody algebras

Following Belinskii, Khalatnikov and Lifshitz we suppose the dynamics of gravity close to a spacelike singularity can be described, at each spatial point, as a piecewise linear motion of an auxiliary particle in a region of hyperbolic space [1, 8, 9]. The BKL analysis uses an

asymptotic spacetime metric of the form

$$ds^2 = -(g d\tau)^2 + \sum_{i=1}^d e^{-2\beta^i(x,\tau)} (\omega^i)^2, \quad (2.1)$$

where g is the determinant of the metric on spatial slices and ω^i is the spatial ‘‘Iwasawa frame’’ (see [10, 59]). The singularity occurs at $\tau \rightarrow \infty$ when the spatial volume density g collapses locally at each spatial point. The equations of motion imply that the dynamics of the gravity-dilaton- p -form system at a fixed spatial point is that of a massless particle with coordinates $\beta^\mu(\tau)$ moving in an auxiliary space, called β -space, \mathfrak{M}_β , with metric

$$d\sigma^2 = G_{\mu\nu} d\beta^\mu d\beta^\nu = \sum_i d\beta^i d\beta^i - \sum_{i,j} d\beta^i d\beta^j + \sum_{k=1}^q d\phi^k d\phi^k. \quad (2.2)$$

This metric is flat and of Lorentzian signature, and so we have

$$\mathfrak{M}_\beta \simeq \mathbb{R}^{1,M-1}, \quad (2.3)$$

where $M = d + q$ (q being the number of dilatons ϕ). The dynamics in the BKL-limit corresponds to free linear motion in \mathfrak{M}_β , interrupted by specular reflections against the walls defined by $\omega_A(\beta) = 0$, where $\omega_A \in \mathfrak{M}_\beta^*$ are the wall forms. The wall forms include the symmetry walls

$$s_{ij}(\beta) = \beta^i - \beta^j, \quad (i, j = 1, \dots, d; i > j), \quad (2.4)$$

the gravitational walls⁴

$$G_{ijk}(\beta) = 2\beta^i + \sum_{m \neq i,j,k} \beta^m, \quad (i \neq j, i \neq k, j \neq k), \quad (2.5)$$

and, if p -form fields are present, they produce electric and magnetic walls

$$\begin{aligned} e_{i_1 \dots i_p}^{[p]}(\beta) &= \beta^{i_1} + \dots + \beta^{i_p} + \frac{1}{2} \lambda(p) \phi, & (i_1 < \dots < i_p), \\ m_{i_1 \dots i_{d-p-1}}^{[p]}(\beta) &= \beta^{i_1} + \dots + \beta^{i_{d-p-1}} - \frac{1}{2} \lambda(p) \phi, & (i_1 < \dots < i_{d-p-1}), \end{aligned} \quad (2.6)$$

where $\lambda(p)$ is the dilaton coupling for the p -form in question. All of the wall forms presented above are spacelike, $(\omega|\omega) > 0$, implying that the associated walls defined by $\omega(\beta) = 0$ are all timelike. Since the wall forms belong to the dual space \mathfrak{M}_β^* , their scalar products are computed with the inverse metric

$$(\omega|\omega') = G^{\mu\nu} \omega_\mu \omega'_\nu = \sum_i \omega_i \omega'_i - \frac{1}{d-1} \left(\sum_i \omega_i \right) \left(\sum_j \omega'_j \right) + \sum_{k=1}^q \omega_{\phi^k} \omega'_{\phi^k}, \quad (2.7)$$

⁴There are actually two types of gravity wall forms; the other set of wall forms being $G_i(\beta) = \sum_{j \neq i} \beta^j$. These are special because they are lightlike, $(G_i|G_i) = 0$, in contrast with all the other wall forms which are spacelike. For the purposes of this paper the walls defined by $G_i(\beta) = 0$ are of no relevance since they are always subdominant with respect to the spacelike gravity walls defined by $G_{ijk}(\beta) = 0$.

The walls confine the billiard motion to the region P_β where all the wall forms present in the case at hand obey $\omega_A(\beta) \geq 0$. These wall forms can be expressed as linear combinations with nonnegative coefficients of a finite number N of “dominant” wall forms ω_s ; hence the billiard region P_β is defined by:

$$P_\beta = \{\beta \in \mathfrak{M}_\beta \mid \omega_s(\beta) \geq 0, s = 1, \dots, N\}. \quad (2.8)$$

The other (subdominant) walls are hidden behind the dominant ones, implying that they are irrelevant in the BKL-limit.

For a given theory, the precise structure of P_β depends on the topology of spacetime because the relevant collection of p -form walls depends itself on the cohomology of the internal manifold in the pre-quantum gravity regime considered here [43]. This will be discussed further in section 2.2.

Because the billiard walls are timelike hyperplanes in $\mathbb{R}^{1,M-1}$ the reflections with respect to the walls preserve the future lightcone $\mathcal{L}^+ \subset \mathbb{R}^{1,M-1}$, and in particular they preserve the set of future-oriented, norm -1 vectors in $\mathbb{R}^{1,M-1}$, corresponding to the hyperbolic space $\mathbb{H}_m \subset \mathcal{L}^+$ ($M = m + 1$).

Not all of the β^μ are physical because of the Hamiltonian constraint

$$\sum_i \partial_\tau \beta^i \partial_\tau \beta^i - \sum_{i,j} \partial_\tau \beta^i \partial_\tau \beta^j + \sum_{k=1}^q \partial_\tau \phi^k \partial_\tau \phi^k = 0. \quad (2.9)$$

This constraint enables us to project onto a subspace of physical dynamical variables, which can be conveniently taken to be precisely the m -dimensional upper sheet of the unit hyperboloid just introduced

$$\mathbb{H}_m = \{\beta \in \mathfrak{M}_\beta \mid (\beta|\beta) = -1 \text{ and } \beta \in \mathcal{L}^+\}. \quad (2.10)$$

The lightlike linear motion in \mathfrak{M}_β then corresponds to geodesics in hyperbolic space. Similarly the walls, being timelike and going through the origin, project radially onto hyperplanes in \mathbb{H}_m (which are at the same time the intersections of the walls with \mathbb{H}_m). The region in \mathbb{H}_m bounded by the (projected) hyperplanes will henceforth be referred to as the *billiard table*. It is customary to go back and forth between the projected description and the original description in Lorentzian space without always making the distinction explicit.

2.1.2 The “uncompactified billiard”

Consider first the case when all the walls are switched on, as it occurs when no internal dimension is compact [43]. This is also the case relevant when the BKL analysis is pushed all the way to the singularity. The billiard region is then the smallest possible one in the sense that the billiard region of all the other cases will contain the billiard region of the uncompactified theory. We shall denote by $\omega_{A'}$ the dominant walls of the uncompactified theory. While the number of dominant walls relevant to the compact cases might not be equal to the dimension M of \mathfrak{M}_β , it turns out that for all theories whose dimensional reduction to three dimensions is described by a symmetric space, the number of dominant walls is equal to M [12, 13, 10].

In this case the dominant walls confine the billiard motion to the region $\mathcal{B}_\beta \subset \mathfrak{M}_\beta$ defined by

$$\mathcal{B}_\beta = \{\beta \in \mathfrak{M}_\beta \mid \omega_{A'}(\beta) \geq 0, A' = 1, \dots, M\}, \quad (2.11)$$

The billiard table is a simplex in \mathbb{H}_m . We shall call somewhat improperly the region (2.11) the “uncompactified billiard region” and its projection on \mathbb{H}_m the “uncompactified billiard table”.

The scalar products $(\omega_{A'}|\omega_{B'})$ between the dominant walls can be organized into a matrix,

$$A_{A'B'} = \frac{2(\omega_{A'}|\omega_{B'})}{(\omega_{A'}|\omega_{A'})}. \quad (2.12)$$

In the noncompact case, the matrix A turns out to possess the properties of a Lorentzian Cartan matrix [12, 10], thereby identifying the dominant wall forms $\omega_{A'}$ with the simple roots of the Kac-Moody algebra $\mathfrak{g}(A)$ constructed from A [17]. This Kac-Moody algebra is the “overextension” \mathfrak{u}^{++} of the U-duality algebra \mathfrak{u} [13]. The group generated by the reflections in the billiard walls of the uncompactified theory is a Coxeter group, which is the Weyl group of the corresponding Kac-Moody algebra [12, 10]. We shall denote it by \mathfrak{W} .

The action of \mathfrak{W} on \mathcal{L}^+ splits up into a disjoint union of chambers, called *Weyl chambers*. One of these chambers is defined by the inequalities $\omega_{A'} \geq 0$ and is called the *fundamental chamber* \mathcal{F} . Then, all other chambers in \mathcal{L}^+ correspond to images of \mathcal{F} under \mathfrak{W} . The action of \mathfrak{W} on the Weyl chambers is simply transitive. When projected onto \mathbb{H}_m , these chambers become m -simplices of finite volume. The fundamental chamber \mathcal{F} is the uncompactified billiard region in which the chaotic dynamics takes place. The $m + 1$ hyperplanes (or dominant walls) which bound the fundamental chamber, correspond to the codimension-one faces of \mathcal{F} when projected onto \mathbb{H}_n (see also section 3.3.1).

2.2 Compactification and cohomology

Compactification can modify the billiard, as was shown in [43, 44]. This occurs because the billiard dynamics in the intermediate regime considered here depends not only on the p -form menu, but also on the topology of the space upon which the theory is formulated, specifically on the de Rham cohomology $\mathcal{H}^p(\mathcal{M})$ of the compactification manifold \mathcal{M} . The rules for constructing the noncompact billiard system are given above, and here we focus on the “selection rule” that describes how this billiard is modified after compactification.

2.2.1 Selection rule

We study situations in which all spatial dimensions are compact, and thus spacetime Σ has topology

$$\Sigma = \mathbb{R} \times \mathcal{M}, \quad (2.13)$$

where \mathcal{M} is closed, compact, and orientable. Electric and magnetic walls, $e(\beta)$ and $m(\beta)$, arise from the electric and magnetic components, F_E and F_M , of a given p -form F . On a compact manifold \mathcal{M} the p -form fields which remain massless during the compactification correspond to solutions of the equations of motion of the form

$$F_E = f_E(t) \omega_p \wedge dt, \quad F_M = f_B(t) \omega_{p+1}, \quad (2.14)$$

where ω_p and ω_{p+1} are representatives of the de Rham cohomology classes $\mathcal{H}^p(\mathcal{M})$ and $\mathcal{H}^{p+1}(\mathcal{M})$, respectively. When \mathcal{M} is compact, solutions that yield electric and magnetic billiard walls can therefore only be found when the de Rham cohomology classes are non-trivial.⁵

We may now state the influence of the topology of \mathcal{M} on the billiard structure simply in terms of a *selection rule*. This rule makes use of the Betti numbers $b_j(\mathcal{M})$, which are the dimensions of the de Rham cohomology classes $\mathcal{H}^p(\mathcal{M})$. The selection rule reads [43, 44]:

- **Selection Rule:** *When $b_s(\mathcal{M}) = 0$ for some s , we remove all billiard walls corresponding to electric s -forms, or magnetic $(s - 1)$ -forms.*

The selection rule is established using the same assumptions as the noncompact BKL analysis, namely that we are in a regime where classical gravity is valid, and studying a sufficiently “generic” spacetime solution.

It has been known for some time that the algebraic structure of the billiard is invariant under Kaluza-Klein reduction on tori [13]. The selection rule is compatible with these results, since tori have no vanishing Betti numbers. Note also that none of the symmetry walls (or gravity walls) is eliminated by the compactification. Hence, among the dominant walls of the compactified theory we always have the $(d - 1)$ dominant symmetry walls $\beta^2 - \beta^1, \dots, \beta^d - \beta^{d-1}$.

2.3 Gram matrices and Coxeter groups

In order to understand the structure of the reflection group that emerges when some p -form walls are switched off as well as the features of the corresponding billiard domain, it is useful to recall a few facts about polyhedra and reflection groups in hyperbolic space. The main reference for this section is [56].

2.3.1 Convex polyhedra

We shall consider convex polyhedra P of hyperbolic space, i.e., regions of the form

$$P = \bigcap_{s=1}^N H_s^+, \tag{2.15}$$

where H_s^+ is a half-space bounded by the hyperplane H_s , and N is the number of such bounding hyperplanes. In our case, H_s is one of the walls of the relevant dominant wall system,

$$H_s = \{\beta \in \mathfrak{M}_\beta \mid \omega_s(\beta) = 0\}, \tag{2.16}$$

⁵For more general compactifications with fluxes turned on, this simple argument does no longer hold because then the massless spectrum is not determined only by the de Rham cohomology of \mathcal{M} . Instead one must consider a more complicated “twisted” cohomology (see, e.g., [60–62]). However, in our analysis, these kinds of compactifications are actually trivial, since all p -forms are lifted and become massive. Hence, the billiard dynamics will be controlled by the purely gravitational sector, or the gravity-dilaton sector if dilatons are present. We thank Larus Thorlacius and Alexander Wijns for raising this issue, and for helpful discussions.

and H_s^+ is defined by

$$H_s^+ = \{\beta \in \mathfrak{M}_\beta \mid \omega_s(\beta) \geq 0\}. \quad (2.17)$$

The polyhedra P therefore contain the fundamental domain (2.11), defined by the dominant walls of the uncompactified theory. Hence it is clear that P has non-vanishing volume.

2.3.2 Relative positions of walls in hyperbolic space

It is customary to associate with the convex polyhedron P a matrix $G(P)$, the so-called *Gram matrix*, which differs from the matrix A by normalization. The construction proceeds as follows. To each hyperplane H_s we associate a unit spacelike vector e_s pointing inside P , i.e., pointing towards the billiard region (which is thus defined by $(\beta, e_s) \geq 0$). We then construct the $N \times N$ matrix $G(P)$ of scalar products $(e_s|e_{s'})$. Four cases can occur for the scalar product $(e_s|e_{s'})$ between a given pair of distinct unit vectors e_s and $e_{s'}$ [56]:⁶

1. $-1 \leq (e_s|e_{s'}) \leq 0$. In this case, the hyperplanes H_s and $H_{s'}$ intersect and form an acute angle. The limiting case $(e_s|e_{s'}) = -1$ means that the hyperplanes intersect at infinity, i.e., are parallel. The other limiting case $(e_s|e_{s'}) = 0$ means that the hyperplanes form a right angle, which is both acute and obtuse.
2. $0 \leq (e_s|e_{s'}) \leq 1$. In this case, the hyperplanes also intersect, but form an obtuse angle. The limiting case $(e_s|e_{s'}) = 1$ corresponds again to parallel hyperplanes meeting at infinity.
3. $(e_s|e_{s'}) < -1$.
4. $(e_s|e_{s'}) > 1$.

In the latter two cases, the hyperplanes diverge, i.e., do not meet even at infinity. The fourth case and the limiting case $(e_s|e_{s'}) = 1$ ($s \neq s'$) are excluded from our analysis, because the geometry defined by the relevant electromagnetic walls has a metric which is positive semi-definite, so that one can have $(e_s|e_{s'}) \geq 1$ only if one of the walls is the (dominant) gravitational wall. But this cannot arise, as shown by our analysis of scalar products involving the gravitational wall in section 3 (taking into account the normalisation of the simple roots).

We denote the dihedral angle between the hyperplanes H_s and $H_{s'}$ by $H_s^+ \cap H_{s'}^+$. When the hyperplanes intersect, the value of the dihedral angle $H_s^+ \cap H_{s'}^+$ can be found from the relation

$$\cos(H_s^+ \cap H_{s'}^+) = -(e_s|e_{s'}). \quad (2.18)$$

2.3.3 Acute-angled polyhedra

If the number of dominant walls is strictly smaller than the dimension M of \mathfrak{M}_β , the billiard table in \mathbb{H}_m has infinite volume and the motion is non-chaotic. After a finite number of collisions, the billiard ball escapes to infinity [6, 57]. We shall therefore assume that the

⁶Note that one cannot have $e_s = -e_{s'}$ since then the region $H_s^+ \cap H_{s'}^+ = H_s$ has vanishing volume, which is excluded as we observed above.

number of dominant walls is greater than or equal to the dimension M of \mathfrak{M}_β , a case that needs a more detailed analysis. The Gram matrix is then of rank M because among the S dominant wall forms, one can find a subset of M of them that defines a basis, namely the $(d - 1)$ symmetry walls and one of the other dominant walls if there is no dilaton (or two linearly independent dominant non-symmetry walls if there is a dilaton etc.). The convex polyhedron defined by the dominant walls is therefore non-degenerate (see [56]). We shall also assume that the Gram matrix is indecomposable (cannot be written as a direct sum upon reordering of the e_i 's), as the decomposable case can be analysed in terms of the indecomposable one.

If the number of dominant walls is exactly equal to M , the billiard table is a simplex in hyperbolic space. Otherwise, one has a non-simplex billiard table, with the number of faces exceeding $\dim \mathbb{H}_n + 1 = n + 1$.

A crucial notion in the study of reflection groups is that of *acute-angled polyhedra*. A convex polyhedron is said to be *acute-angled* if for any pair of distinct hyperplanes defining it, either the hyperplanes do not intersect, or, if they do, the dihedral angle $H_s^+ \cap H_{s'}^+$ does not exceed $\frac{\pi}{2}$. The Gram matrix, which has 1's on the diagonal, has then negative entries off the diagonal. While non-degenerate, indecomposable acute-angled polyhedra on the sphere or on the plane are necessarily simplices, this is not the case on hyperbolic space.

2.3.4 Coxeter polyhedra and the billiard group

We have seen that the dynamics of gravity is described in all cases (uncompactified or compactified) by the motion of a billiard ball in a region of hyperbolic space. The reflections s_s ($s = 1, \dots, N$) with respect to the billiard walls generate a discrete reflection group which we want to characterize. This group, which we shall call the *billiard group* and denote by \mathfrak{B} , is a subgroup of the Coxeter group relevant to the uncompactified case, where the total number of walls is maximum and the billiard region the smallest (and contained in all other billiard regions). The billiard group is a crystallographic Coxeter group since it is generated by reflections [56] and since it preserves the root lattice of the Kac-Moody algebra of the uncompactified case. Its presentation in terms of the billiard walls might, however, be non-standard. The billiard group \mathfrak{B} will be examined more carefully in sections 3.2.2 and 3.2.3.

The billiard table has an important property which it inherits from the complete wall system of the theory. Consider the dihedral angle $H_i^+ \cap H_j^+$ between two different walls H_i and H_j that intersect. If this angle is acute, then it is an integer submultiple of π , i.e., of the form π/m_{ij} where m_{ij} is an integer greater than or equal to 2. If this angle is obtuse, then $\pi - H_i^+ \cap H_j^+$ is an integer submultiple of π , i.e., $\pi - H_i^+ \cap H_j^+ = \pi/m_{ij}$, where $m_{ij} \in \mathbb{Z}_{\geq 2}$. If the walls do not intersect, they are parallel and one has $m_{ij} = \infty$. Thus, given any pair of distinct walls, one can associate to it an integer $m_{ij} = m_{ji} \geq 2$.

In the case when all the angles are acute, and hence integer submultiples of π , the polyhedron is called a *Coxeter polyhedron*. Coxeter polyhedra are thus acute-angled. In hyperbolic space, they may or may not be simplices.

3. General results

In this section we describe our new results concerning general features of the billiard structures after compactification. In section 3.1, we use features of the wall system to explain why the billiard table need not be a Coxeter polyhedron after compactification. In section 3.3 we describe the billiard region after compactification in terms of galleries, which we explain. Finally, we describe in section 3.4 our methods for determining the chaotic properties for all possible compactifications.

3.1 Rules of the game

We show why the billiard region need not be a Coxeter polyhedron after compactification with the aid of two facts about billiard wall systems:

- **Fact 1:** *If both an electric and a magnetic wall are present for a given p -form, then the gravitational walls are subdominant.* This was proven in [38, 13], by noticing that for any p we have

$$G_{ijk} = e_{r_1 \dots r_p}^{[p]} + m_{s_1 \dots s_{d-p-1}}^{[p]}, \tag{3.1}$$

where one of the r_q and one of the s_q are equal to i , and neither j nor k appears in either the r_q or s_q .

- **Fact 2:** *The inner product between a dominant gravitational wall and a dominant p -form wall is unity for*
 - *electric walls with $p \leq D - 3$,*
 - *magnetic walls with $p \geq 1$,*

and the inner product vanishes for

- *electric walls with $p = D - 2$,*
- *magnetic walls with $p = 0$.*

Typically, p forms with $p > \lfloor D/2 \rfloor - 1$ are dualised so that we may safely assume that $p \leq \lfloor D/2 \rfloor - 1$. (This is a stronger condition than $p \leq D - 3$ for $D \geq 4$). Then, we have that the inner product between gravitational and p -form walls is always positive, except when the p -form is a scalar (axion) and the inner product vanishes. This fact is proven by computing the relevant inner products (which are independent of the dilaton coupling(s) of the p -form) using the metric between wall forms.

This fact is significant in combination with the requirement that a system of dominant walls define an acute-angled polyhedron if the inner products between each pair of walls are either zero or negative. Therefore we can conclude:

If the dominant wall set includes a gravitational wall and any non-scalar p -form wall, then the dominant wall system does not define a Coxeter polyhedron.

For compactifications on tori, only **Fact 1** is relevant since all components of the same p -forms are preserved in the compactification [13]. Hence, electric and magnetic walls are always present in pairs, ensuring that the gravity wall is always subdominant. More general compactifications can eliminate one of the electric or magnetic walls of a given p -form, while leaving the other intact. Unlike the noncompact case, it is possible for both gravitational and p -form walls to appear simultaneously in the set of dominant walls. **Fact 2** tells us that when this happens, we no longer have a Coxeter polyhedron.

In our analysis, we never eliminate gravitational walls (although there are some partial results regarding their selection rules that were given in [43]). This means that after compactification there is always a gravitational wall in the root system, though it is not necessarily dominant. Therefore all compactifications we study fall into one of three classes:

- A pair of electric and magnetic walls from the noncompact theory has not been eliminated by compactification, so the gravitational wall is subdominant. The resulting billiard table may be a Coxeter polyhedron, depending on the details of the p -form menu and couplings in the theory. (Example: the $b_1 = 0$ compactification of the B_n sequence, see section 4.)
- One (or both) members of each pair of electric/magnetic walls are eliminated, so the gravitational wall is exposed. If there are any other p -form walls left, because of **Fact 2**, the billiard table cannot be a Coxeter polyhedron. (Example: the $b_3 = 0$ compactification in the E_8 sequence.) However, it may occur that “coincidentally” two walls from different p -forms can combine and make the gravitational wall subdominant.
- All of the p -form walls are eliminated by compactification, and so only the gravitational wall is left. In this case one always obtains A_n^{++} , the billiard corresponding to pure gravity. This billiard sits at the bottom of every list when all possible Betti numbers are set to zero. Occasionally there are also direct summands corresponding to scalar fields that are never eliminated by compactification. (Example: $b_1 = b_2 = 0$ for C_n , with $n > 4$, as a billiard described by $A_1^{++} \times C_{n-1}$.)

Only the first of the cases described above arises in the noncompact theory, where gravitational walls are never dominant, except when they are the only non-symmetry walls.

3.2 Describing the billiard group after compactification

We shall now begin to assemble the various structures described so far in order to analyze the group-theoretical properties of the billiard dynamics after compactification. This involves understanding the relation between the billiard table and the fundamental domain of the associated reflection group. In this context it is important to distinguish between the formal Coxeter group and the billiard group. We consider these concepts in turn in the following two sections, and explain how they are related in section 3.2.3.

3.2.1 The formal Coxeter group

Recall from section 2.3.4 that the reflections s_i with respect to the dominant walls H_i generate a Coxeter group. To describe this group we must characterize the relations among

the reflections s_i . Being reflections, they clearly satisfy

$$s_i^2 = 1. \tag{3.2}$$

Consider next the reflections s_i and s_j with respect to two different hyperplanes H_i and H_j . Then, the product $s_i s_j$ is a rotation by the angle $\frac{2\pi}{m_{ij}}$ (where the integers m_{ij} were introduced in section 2.3.4) and so

$$(s_i s_j)^{m_{ij}} = 1. \tag{3.3}$$

These relations alone define a Coxeter group, which we shall call the *formal Coxeter group* \mathfrak{C} associated with the billiard. One can describe \mathfrak{C} more precisely as follows. Let $\tilde{\mathfrak{C}}$ be the group freely generated by the elements of the set $\mathcal{S} = \{r_1, \dots, r_N\}$, and let \mathfrak{N} be the normal subgroup generated by $(r_i r_j)^{m_{ij}}$, where the *Coxeter exponents* m_{ij} satisfy [17, 58] (see also [59])

$$\begin{aligned} m_{ii} &= 1, \\ m_{ij} &\in \mathbb{Z}_{\geq 2}, \quad i \neq j, \\ m_{ij} &= m_{ji}. \end{aligned} \tag{3.4}$$

Then \mathfrak{C} is defined as the quotient group $\tilde{\mathfrak{C}}/\mathfrak{N}$ and has the following standard presentation:

$$\mathfrak{C} = \langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1, \quad i, j = 1, \dots, N \rangle. \tag{3.5}$$

One can associate a Coxeter graph with the formal Coxeter group, i.e., with the set of m_{ij} 's. Each r_i defines a node of the Coxeter graph, and two different nodes i and j are connected by a line whenever $m_{ij} > 2$, with m_{ij} explicitly written over the line whenever $m_{ij} > 3$ [58]. To a Coxeter graph, one can further associate a symmetric matrix defined by

$$B_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right) \tag{3.6}$$

with 1's on the diagonal and non-positive elements otherwise.

3.2.2 The billiard group and its fundamental domain

In section 2.3.4 we introduced the concept of the *billiard group* \mathfrak{B} . Here we shall elaborate on this object and elucidate the structure of its fundamental domain.

The billiard group \mathfrak{B} is defined as the group generated by the reflections s_i with respect to the dominant walls of the billiard table. This group coincides with the formal Coxeter group \mathfrak{C} (with $r_i \equiv s_i$) if and only if there are no additional relations among the s_i 's. Two cases must be considered.

1. **Acute-angled billiard tables.** There is no additional relations among the s_i 's if and only if the billiard table is acute-angled, i.e., is a Coxeter polyhedron [56]. In that case, the matrix $B = (B_{ij})$ coincides with the Gram matrix $G(P)$. Furthermore, the billiard table is a fundamental domain [56].

In the hyperbolic case relevant here, the billiard table need not be a simplex. When it is a simplex, however, there is further structure. The Gram matrix is non-degenerate and the action of the Coxeter group on the β -space \mathfrak{M}_β coincides with the standard geometric realization considered in [58]. In addition, the matrix

$$A_{ss'} = \frac{2(\omega_s|\omega_{s'})}{(\omega_s|\omega_s)}, \tag{3.7}$$

has non-positive integers off the diagonal and hence is a non-degenerate Cartan matrix. It is obvious that the off-diagonal entries are integers since the walls correspond to some roots of the Kac-Moody algebra of the uncompactified case. Moreover, it follows from the fact that the billiard table is acute-angled that they are negative. The billiard group is then the Weyl group of a simple Kac-Moody algebra.⁷ When the matrix A is a Cartan matrix, one can associate to it a Dynkin diagram.

2. **Non acute-angled billiard tables.** If the billiard table is not acute-angled (and hence not a Coxeter polyhedron), there are further relations among the s_i 's and the billiard group is therefore the quotient of the formal Coxeter group \mathfrak{C} by these additional relations. Moreover, the billiard table is not a fundamental domain of the billiard group. One may understand this feature as follows. Consider a dihedral angle $H_i^+ \cap H_j^+$ of the polyhedron that is obtuse. The rotation $s_i s_j$ by the angle $2\pi/m_{ij}$ is an element of the group and hence maps reflection hyperplanes to reflection hyperplanes. The image by this rotation of H_j is the hyperplane $s_i H_j$ that intersects the interior of the billiard table.⁸ The reflection $s_i s_j s_i$ with respect to this hyperplane belongs to the group, and hence the billiard table cannot be a fundamental region of the billiard group since the orbit of a point sufficiently close to $s_i H_j$ intersects the billiard table at least twice. Fundamental regions are obtained by considering all the mirrors (reflection hyperplanes) associated with the group (most of which are not billiard walls), which decompose the space into equivalent chambers that are permuted by the group (homogeneous decomposition). Each of these chambers is a fundamental domain. The billiard group \mathfrak{B} is generated by the reflections in the mirrors of the fundamental domain, which provide a standard presentation of the group, and the billiard table is a union of chambers [56]. Examples of the occurrence of this phenomenon will be discussed below.

Although the billiard table is not a fundamental domain, it can be naturally described as a gallery defined by the Coxeter group of the uncompactified theory. This is described in section 3.3.2.

⁷In the case of a non-simplex, acute-angled billiard table, the matrix A is also a valid Cartan matrix, but it is degenerate and the corresponding Kac-Moody algebra is not simple. Furthermore, the standard geometric realization is defined in a space of dimension larger than M , while the billiard realization is defined in the M -dimensional β -space.

⁸When the Coxeter group is crystallographic ($m_{ij} \in \{2, 3, 4, 6\}$), the converse is also true: if the angle between H_i and H_j is acute, then the image $s_i H_j$ does *not* intersect the interior of the billiard table.

3.2.3 Non-standard presentations of Coxeter groups

We have seen how one can associate a formal Coxeter group \mathfrak{C} to the billiard region using the Coxeter presentation $\mathfrak{C} = \tilde{\mathfrak{C}}/\mathfrak{N}$. The billiard group \mathfrak{B} — which is also a Coxeter group — differs from \mathfrak{C} when the billiard table possesses obtuse angles because the reflections in the walls of the billiard then fulfill additional relations. This yields a non-standard presentation of the billiard group, which can be formally described as follows.

Let \mathcal{B}_β be the billiard region after compactification, let the elements of the set $\mathcal{S} = \{s_i \mid i = 1, \dots, N\}$ be the reflections in the walls W_i bounding \mathcal{B}_β , and let $\tilde{\mathfrak{C}}$ be the formal group freely generated by \mathcal{S} . The dihedral angles between the W_i give rise to a set of Coxeter exponents m_{ij} , with associated normal subgroup $\mathfrak{N} \subset \tilde{\mathfrak{C}}$. Suppose now that the region \mathcal{B}_β is not a fundamental domain of \mathfrak{B} , and denote by \mathfrak{J} the normal subgroup of $\tilde{\mathfrak{C}}$ generated by $(s_i s_j)^{m_{ij}}$ and any other non-standard relations between the elements of \mathcal{S} . Note that we have $\mathfrak{N} \subset \mathfrak{J}$. The billiard group \mathfrak{B} is then the quotient

$$\mathfrak{B} = \tilde{\mathfrak{C}}/\mathfrak{J}. \tag{3.8}$$

Equivalently, if we denote by \mathfrak{F} the normal subgroup of the formal Coxeter group \mathfrak{C} generated only by the non-standard relations between the elements of \mathcal{S} , we may describe the billiard group as

$$\mathfrak{B} = \mathfrak{C}/\mathfrak{F}. \tag{3.9}$$

Neither of these presentations is a Coxeter presentation.

In all cases we consider in this paper, the uncompactified billiard is described by the Weyl group $\mathfrak{W}[\mathfrak{u}^{++}]$ of a Lorentzian Kac-Moody algebra \mathfrak{u}^{++} . For general compactifications, the billiard group \mathfrak{B} is therefore a Coxeter subgroup of $\mathfrak{W}[\mathfrak{u}^{++}]$. However, we shall also see examples of cases when the formal Coxeter group \mathfrak{C} after compactification differs from \mathfrak{u}^{++} , while the billiard group \mathfrak{B} is actually isomorphic \mathfrak{u}^{++} , with a non-standard presentation. This is described in detail in section 4.2.

3.3 Fundamental domains, chamber complexes and galleries

We have seen that the billiard table need not be a Coxeter polyhedron upon compactification. When it is not a Coxeter polyhedron, it no longer corresponds a fundamental domain of the billiard group \mathfrak{B} . Moreover, \mathfrak{B} is the quotient by nontrivial extra relations of the formal Coxeter group associated with the billiard table. In this section, we describe how the billiard region relevant to the compactified case can then be built as a union of images of the uncompactified billiard region. This is achieved using the theory of buildings, in terms of chambers and galleries.

3.3.1 Chambers and galleries

The analysis in this section makes use of the treatment of Coxeter groups as a theory of *buildings*, a formalism mainly developed by J. Tits. Introductions and references may be found in [63, 64].

The basic idea is to study Coxeter groups in geometric language by defining them in terms of the objects on which they act nicely. The *buildings* are the fundamental objects

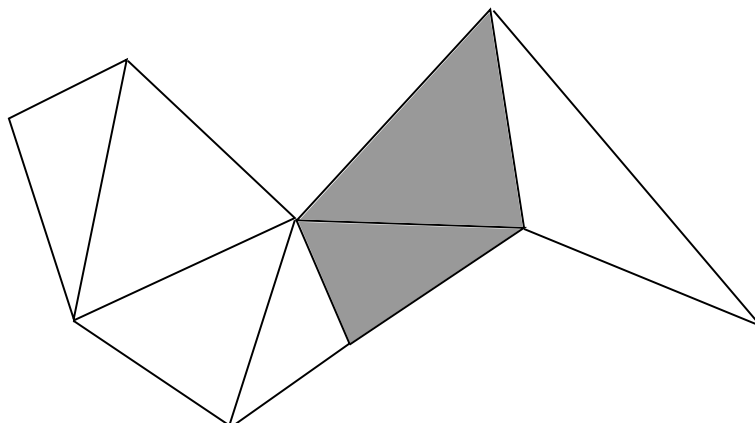


Figure 1: A simplicial complex of 2-simplices (triangles). The two shaded regions represent adjacent (maximal) simplices.

which then defines the associated Coxeter group. For example, finite Coxeter groups act on so-called *spherical buildings*, because these groups preserve the unit sphere. We are interested in Coxeter groups which act on *hyperbolic buildings*, i.e., hyperbolic Coxeter groups, which preserve the hyperbolic space.

An n -simplex \mathcal{X} in hyperbolic space is determined by its $n + 1$ vertices. A 1-simplex is determined by its two endpoints, a 2-simplex (a triangle) is determined by its three vertices etc. For this reason, it is convenient to identify an n -simplex \mathcal{X} with the set of its $n + 1$ vertices, $\mathcal{X} = \{\text{set of } n + 1 \text{ vertices}\}$. A *face* f of \mathcal{X} is a simplex corresponding to a non-empty subset $f \subset \mathcal{X}$. The *codimension* of f with respect to \mathcal{X} is given by $\dim \mathcal{X} - \dim f$. For example, there are three codimension-one faces in a 2-simplex, which are the three edges of the triangle.

Next we define the notion of a simplicial complex. Let V be a set of vertices, and \mathcal{K} a set of finite subsets $\mathcal{X}_k \subset V$. We assume that the subsets containing a single vertex of V are all elements of \mathcal{K} . Then \mathcal{K} is called a *simplicial complex* if it is such that given $\mathcal{X} \in \mathcal{K}$ and a face f of \mathcal{X} , then $f \in \mathcal{K}$. The elements \mathcal{X}_k of \mathcal{K} are the simplices in the simplicial complex. Two simplices \mathcal{X}_1 and \mathcal{X}_2 of the same dimension in \mathcal{K} are called *adjacent* if they share a codimension-one face, i.e., if they are separated by a common wall. Figure 1 illustrates a simplicial complex of 2-simplices.

A *maximal simplex* \mathcal{C} in \mathcal{K} is such that \mathcal{C} does not correspond to the face of another simplex in \mathcal{K} . Maximal simplices in a simplicial complex are called *chambers* and they shall be our main objects of study. A sequence of chambers $\mathcal{C}_1, \dots, \mathcal{C}_k$, such that any two consecutive chambers \mathcal{C}_i and \mathcal{C}_{i+1} are adjacent is called a *gallery* Γ . Thus, a gallery corresponds to a connected path between two chambers \mathcal{C}_1 and \mathcal{C}_k in \mathcal{K} , and we write

$$\Gamma : \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{k-1}, \mathcal{C}_k. \tag{3.10}$$

The *length* of Γ is k , and the *distance* between \mathcal{C}_1 and \mathcal{C}_k is the length of the shortest gallery connecting them. If any two chambers in \mathcal{K} are connected by a gallery, then the

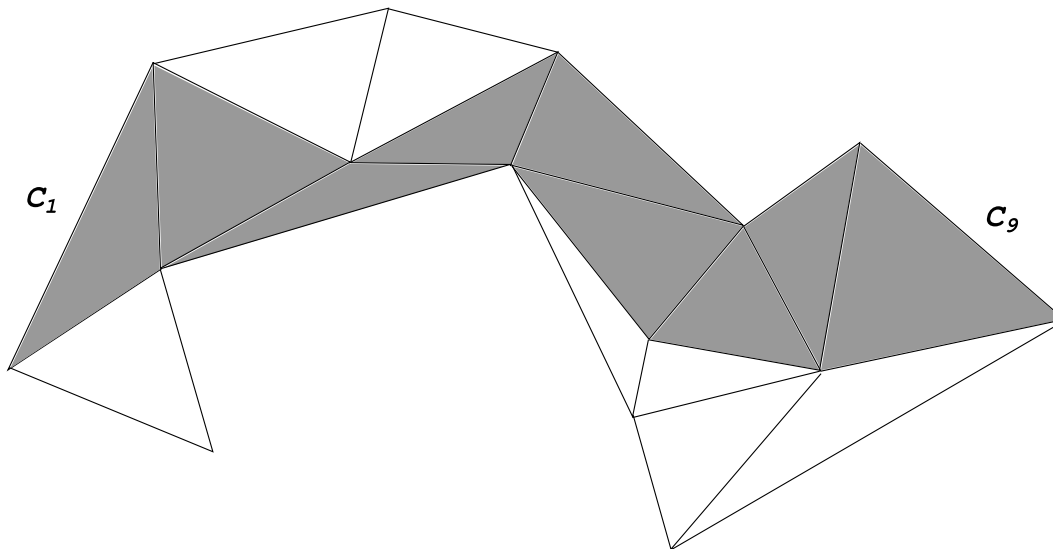


Figure 2: A chamber complex with a gallery $\Gamma : \mathcal{C}_1, \dots, \mathcal{C}_9$, represented by the shaded region. The length of the gallery is $k = 9$, which is also the distance between \mathcal{C}_1 and \mathcal{C}_9 since Γ is the shortest gallery connecting \mathcal{C}_1 and \mathcal{C}_9 .

simplicial complex is called a *chamber complex*. A simple example of a gallery inside a chamber complex is displayed in figure 2.

3.3.2 The billiard region as gallery

We describe the billiard regions of compactified gravity-dilaton- p -form theories in the language presented above. This is achieved by expressing them in terms of the billiard region of the uncompactified theory (or compactified on a torus) which we recall is the fundamental Weyl chamber \mathcal{F} of the Weyl group $\mathfrak{W}[\mathfrak{u}^{++}]$ of the Kac-Moody algebra \mathfrak{u}^{++} , whose Cartan matrix is defined through the scalar products between the dominant wall forms.

Compactification amounts to a process of removing dominant walls, and so the billiard table is enlarged. The inequalities $\omega_{A'} \geq 0$ associated with the simple roots of the underlying Kac-Moody algebra are indeed replaced by weaker inequalities. The bigger region so defined can be written as a union of Weyl chambers of $\mathfrak{W}[\mathfrak{u}^{++}]$. We shall illustrate this phenomenon on the example of the familiar Lie algebra A_3 , whose Weyl group is a spherical Coxeter group. The fundamental Weyl chamber \mathcal{F} is defined by

$$\omega_1(\beta) \geq 0, \quad \omega_2(\beta) \geq 0, \quad \omega_3(\beta) \geq 0, \tag{3.11}$$

corresponding to the three simple roots. The non simple roots are $\omega_1 + \omega_2$, $\omega_2 + \omega_3$ and $\omega_1 + \omega_2 + \omega_3$. Suppose that the single dominant wall W_1 defined by $\omega_1(\beta) = 0$ is suppressed. Effectively, this implies that the particle geodesic may cross the wall W_1 . Thus it moves from the region where $\omega_1(\beta) \geq 0$ into the region where

$$\omega_1(\beta) \leq 0. \tag{3.12}$$

We shall consider two cases. (i) The new billiard region is defined by

$$\omega_1(\beta) + \omega_2(\beta) \geq 0, \quad \omega_2(\beta) \geq 0, \quad \omega_3(\beta) \geq 0, \tag{3.13}$$

i.e., the wall W_1 is replaced by the wall $\omega_1(\beta) + \omega_2(\beta) = 0$; (ii) In the second case, we suppose that also the wall defined by $\omega_1(\beta) + \omega_2(\beta) = 0$ is suppressed. Then the new billiard region is defined by the inequalities

$$\omega_1(\beta) + \omega_2(\beta) + \omega_3(\beta) \geq 0, \quad \omega_2(\beta) \geq 0, \quad \omega_3(\beta) \geq 0 \quad (3.14)$$

i.e., the wall W_1 is replaced by the wall $\omega_1(\beta) + \omega_2(\beta) + \omega_3(\beta) = 0$.

In the first case, one can write the new billiard region as the union $\mathcal{F} \cup \mathcal{A}_2$ where \mathcal{A}_2 is defined by the inequalities

$$\omega_1(\beta) \leq 0, \quad \omega_1(\beta) + \omega_2(\beta) \geq 0, \quad \omega_3(\beta) \geq 0 \quad (3.15)$$

(which imply $\omega_2(\beta) \geq 0$). The region \mathcal{A}_2 is the Weyl chamber obtained by reflecting the fundamental Weyl chamber across the wall W_1 since

$$\begin{aligned} s_1(\omega_1) &= -\omega_1, \\ s_1(\omega_2) &= \omega_1 + \omega_2, \\ s_1(\omega_3) &= \omega_3. \end{aligned} \quad (3.16)$$

Hence,

$$s_1 \cdot \mathcal{F} = \{\beta \in \mathfrak{h} \mid \omega_1(\beta) \leq 0, (\omega_1 + \omega_2)(\beta) \geq 0, \omega_3(\beta) \geq 0\} = \mathcal{A}_2. \quad (3.17)$$

A reflection of this type which maps a chamber \mathcal{C} to an adjacent chamber \mathcal{C}' is known as an *adjacency transformation* [56]. By removing the single dominant W_1 , we therefore get in the first case a new region which is precisely twice as large as the original fundamental region.

In the second case, although we also remove a single dominant wall of the original billiard, we get a larger region. This is because we also remove the subdominant wall $\omega_1(\beta) + \omega_2(\beta) = 0$ which is exposed once W_1 is removed. Indeed, the new billiard region can now be written as the union $\mathcal{F} \cup \mathcal{A}_2 \cup \mathcal{A}_3$ where \mathcal{A}_3 is defined by the inequalities

$$\omega_2(\beta) \geq 0, \quad \omega_1(\beta) + \omega_2(\beta) \leq 0, \quad \omega_1(\beta) + \omega_2(\beta) + \omega_3(\beta) \geq 0 \quad (3.18)$$

(which imply $\omega_1(\beta) \leq 0$ and $\omega_3(\beta) \geq 0$). The region \mathcal{A}_3 is again a Weyl chamber, obtained from \mathcal{A}_2 by acting with the reflection s with respect to $\omega_1 + \omega_2$ since

$$\begin{aligned} s(-\omega_1) &= \omega_2 \\ s(\omega_1 + \omega_2) &= -(\omega_1 + \omega_2), \\ s(\omega_3) &= \omega_1 + \omega_2 + \omega_3. \end{aligned} \quad (3.19)$$

Hence,

$$s \cdot \mathcal{A}_2 = \{\beta \in \mathfrak{h} \mid \omega_2(\beta) \geq 0, (\omega_1 + \omega_2)(\beta) \leq 0, \omega_1(\beta) + \omega_2(\beta) + \omega_3(\beta) \geq 0\} = \mathcal{A}_3. \quad (3.20)$$

Note that \mathcal{A}_2 and \mathcal{A}_3 are adjacent. Thus, while in the second case we also remove a single dominant wall, we now obtain a region three times larger than the fundamental

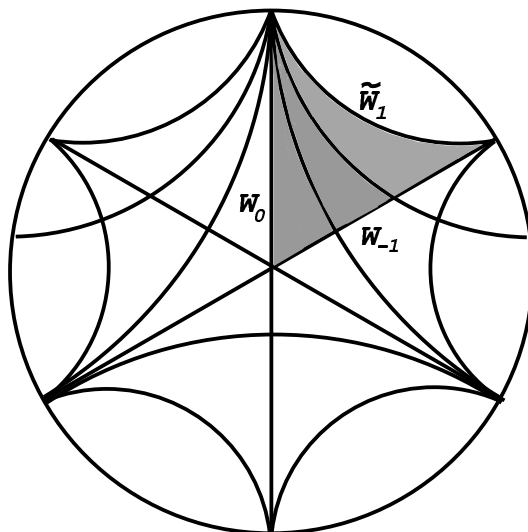


Figure 3: Here we illustrate a gallery $\Gamma : \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of length 3 for the case of the hyperbolic Kac-Moody algebra A_1^{++} . The two walls W_0 and W_{-1} are associated with the affine and overextended simple roots α_0 and α_{-1} , respectively. The original fundamental Weyl chamber $\mathcal{C}_1 = \{h \in \mathfrak{h} \mid \alpha_1(h) \geq 0, \alpha_0(h) \geq 0, \alpha_{-1}(h) \geq 0\}$ corresponds to the leftmost shaded region. We have removed the wall $W_1 = \{h \in \mathfrak{h} \mid \alpha_1(h) = 0\}$ as well as the wall $(\alpha_0 + \alpha_1)(h) = 0$. The far end of the billiard region is now bounded by the new wall $\tilde{W}_1 = \{h \in \mathfrak{h} \mid (2\alpha_0 + 3\alpha_1)(h) = 0\}$. Each of the three chambers is clearly a copy of the fundamental chamber, and the total region is of finite volume. See, e.g., [65, 59] for more detailed discussions of the Weyl group of A_1^{++} .

Weyl chamber. This new region can be described as a union of three pairwise adjacent Weyl chambers. In figure 3 we describe pictorially a similar example for the case of the hyperbolic Coxeter group A_1^{++} .

By extrapolating this analysis to the general case where we remove an arbitrary number $r \leq n + 1$ of dominant walls, we may conclude that the new billiard region corresponds to a union of images of the fundamental Weyl chamber. This naturally has the structure of a simplicial complex, and moreover, by a suitable ordering of the chambers, one sees that it corresponds to a gallery covering the whole complex.⁹ In conclusion, we have found the following: *the billiard region \mathcal{B} obtained by compactification on a manifold of non-trivial topology is described by a gallery Γ inside the Cartan subalgebra \mathfrak{h} of the original hyperbolic Kac-Moody algebra \mathfrak{g} .*

The dynamics after compactification is chaotic if the new billiard region is a finite union of images of the fundamental chamber, i.e., if the gallery Γ has finite length, while if this union is infinite the particle motion will eventually settle down in a single asymptotic Kasner solution, and chaos is removed. Since the Coxeter reflections preserve the volume, the volume of \mathcal{B} is

$$\text{vol } \mathcal{B} = k \cdot \text{vol } \mathcal{F}, \tag{3.21}$$

where k is the length of the gallery Γ associated with \mathcal{B} .

⁹A similar gallery-type structure was recently uncovered in a very different physical context in [66].

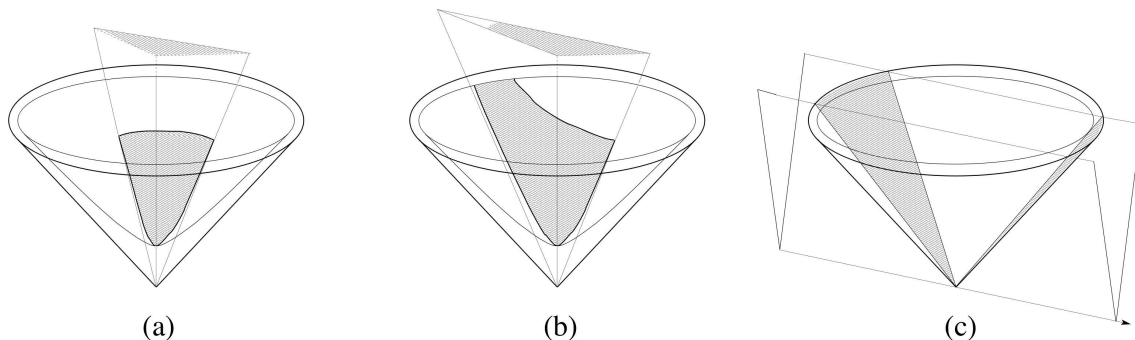


Figure 4: Some examples of the wall systems and their chaotic properties: (a) the wall system corresponding to a hyperbolic Kac-Moody algebra, (b) the wall system of a non-hyperbolic Kac-Moody algebra, or one for which the coweight construction is possible, (c) a wall system with fewer walls than the dimension of the β -space.

3.4 Determining the chaotic properties after compactification

The selection rules described in section 2 provide a straightforward means to determine the billiard system after compactification. Determining whether or not this billiard system is chaotic, i.e., computing the billiard table volume, is somewhat more involved because finding explicitly the corresponding galleries might be intricate. In most cases it is possible to answer this question purely analytically without working out the gallery, although there are several different techniques that work for different types of billiard system. In this section we describe the various methods we employ.

The simplest case is when the billiard table is a Coxeter simplex. The matrix \bar{A}_{ab} is then a Cartan matrix. The associated Kac-Moody algebra $\mathfrak{g}(\bar{A})$ is by construction a regular subalgebra¹⁰ of the Kac-Moody algebra $\mathfrak{g}(A)$, whose Weyl group controlled the uncompactified billiard. The dynamics of the compactified billiard is described by the Weyl group $\bar{\mathfrak{W}}$ of $\mathfrak{g}(\bar{A})$, and the billiard region $\bar{\mathcal{B}}$ coincides with the fundamental domain $\bar{\mathcal{F}}$ of $\bar{\mathfrak{W}}$. Thus, if $\mathfrak{g}(\bar{A})$ is hyperbolic, then $\bar{\mathcal{B}}$ is of finite volume, yielding chaotic dynamics. If the Kac-Moody algebra $\mathfrak{g}(\bar{A})$ is Lorentzian but not hyperbolic, then the billiard is non-chaotic. This is illustrated in figure 4.

In many cases the billiard table is a simplex, but some dihedral angles are obtuse (positive inner product between two different walls) and the matrix A_{ab} is not a proper Cartan matrix. It is however non-degenerate, so that it is possible to define a set of dominant “coweights” $\Lambda^{A'\mu}$ such that

$$\omega_{A'\mu} \Lambda^{B'\mu} = \delta^{B'}_{A'}, \tag{3.22}$$

where ω is a dominant wall labelled by A' . As in the standard Kac-Moody algebra case, these “coweights” span the space of rays that lie within the wall cone, provided we only

¹⁰A subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$ is *regularly embedded* in \mathfrak{g} if and only if two conditions are fulfilled: (i) the root vectors of $\bar{\mathfrak{g}}$ are root vectors of \mathfrak{g} ; and (ii) the simple roots of $\bar{\mathfrak{g}}$ are real roots of \mathfrak{g} . Moreover, the embedding is *positive regular* if the positive root vectors of $\bar{\mathfrak{g}}$ are positive root vectors of \mathfrak{g} . See, e.g., [65, 67] for more detailed discussions on regular subalgebras of Kac-Moody algebras.

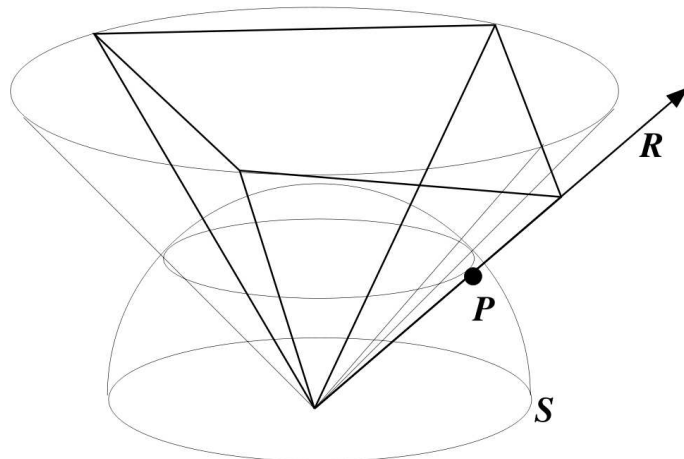


Figure 5: The figure illustrates a non-simplex billiard table. To determine whether the theory is chaotic, it is sufficient to locate a spacelike ray. This is equivalent to maximising the Lorentzian norm on the unit sphere surrounding the origin.

combine “coweights” using non-negative coefficients. A non-chaotic solution to the equations of motion, which corresponds to a null ray within the wall cone, exists if and only if there is at least one timelike and one spacelike “coweight”. This condition is readily checked once the “coweights” are in hand. This technique was employed in [44].

When the billiard table is not a simplex and the number of walls is less than the dimension of \mathfrak{M}_β , then the theory is not chaotic, essentially because there are too few walls to prevent a ray from reaching infinity.

When the billiard table is not a simplex and the number of walls is greater than the dimension of \mathfrak{M}_β , the analysis becomes more complex. This situation is illustrated in figure 5. One method to determine whether chaos is present is to successively remove dominant walls until the billiard region is again a simplex. If there is (at least) one way to do this such that the resulting structure is hyperbolic, then we can conclude that the full region is of finite volume, since reinserting the walls that were removed can never render the volume infinite. In a small number of cases, all wall removals lead to non-chaotic billiards and one cannot conclude immediately whether or not the volume of the billiard table is finite. Another method is then to search numerically for whether a spacelike direction in the wall cone exists. We do this by maximising the Lorentzian norm of points on the unit sphere that lie within the wall cone. If the maximal norm is negative, then no spacelike direction exists and the system is chaotic.

Also, as we have described in section 3.3.1 and 4.2, it is sometimes possible to easily compute the volume of the billiard region exactly by making use of certain properties of the Weyl group $\mathfrak{W}[\mathfrak{g}]$, associated with the uncompactified theory and construct the associated gallery.

As a byproduct of the “coweight construction” mentioned above, we can easily prove the following useful fact:

- **Fact 3:** *Whenever the billiard is described by a direct product $\mathfrak{B}_{fn} \times \mathfrak{B}_{hyp}$ of a finite and a hyperbolic Coxeter group, then the dynamics is non-chaotic.*

This follows by noting that the metric is a direct sum of the metric of \mathfrak{B}_{fin} and that of \mathfrak{B}_{hyp} so the coweights associated with each factor define orthogonal subspaces. Since the coweights associated with the finite factor are spacelike, there will always exist at least one spacelike intersection in the region inside the dominant walls. A different intuitive way to see this is to recall that the volume of the fundamental Weyl chamber of the first factor is finite after projection on the sphere and that of the second factor after projection on the hyperboloid. Two projections are needed to have a finite volume but there is only one here (on a hyperboloid living in the product space).

4. Selected examples

One of the main achievements of this paper is a complete list of billiard structures for general compactifications of all maximally-oxidised theories whose U-duality groups are split real forms. This is presented in full in appendix A, while in this section we present detailed calculations for a selection of these cases in an effort to illuminate the general results of the previous section. We have chosen examples that highlight some particularly interesting or subtle issues in the analysis.

4.1 The F_4 -sequence

For the theory with U-duality algebra F_4 , the Kac-Moody algebraic structure is completely preserved by compactification. The oxidation endpoint of this theory is $D = 6$ with field content given by two scalar fields, a dilaton ϕ and an axion χ , two one-forms A^\pm , a two-form B and, finally, a two-form C with self-dual field strength $G = \star G$. The complete set of walls was found in [13] and is reproduced below.¹¹ We have the following electric walls:

$$\begin{aligned} e_{ij}^{[B]}(\beta) &= \beta^i + \beta^j - \phi, & e_i^{[C]}(\beta) &= \beta^i + \beta^j, \\ e^{[\chi]}(\beta) &= \phi, & e_i^{[A^\pm]}(\beta) &= \beta^i \pm \frac{1}{2}\phi, \end{aligned} \tag{4.1}$$

and magnetic walls:

$$\begin{aligned} m_{ij}^{[B]}(\beta) &= \beta^i + \beta^j + \phi, & m_{ij}^{[C]}(\beta) &= \beta^i + \beta^j, \\ m_{ijkl}^{[\chi]}(\beta) &= \beta^i + \beta^j + \beta^k + \beta^l - \phi, & m_{ijk}^{[A^\pm]}(\beta) &= \beta^i + \beta^j + \beta^k \mp \frac{1}{2}\phi. \end{aligned} \tag{4.2}$$

The subset of dominant walls is

$$\begin{aligned} \alpha_{-1}(\beta) &\equiv s_{54}(\beta) = \beta^5 - \beta^4, & \alpha_0(\beta) &\equiv s_{43}(\beta) = \beta^4 - \beta^3, \\ \alpha_1(\beta) &\equiv s_{32}(\beta) = \beta^3 - \beta^2, & \alpha_2(\beta) &\equiv s_{21}(\beta) = \beta^2 - \beta^1, \\ \alpha_3(\beta) &\equiv e_1^{[A^-]}(\beta) = \beta^1 - \frac{1}{2}\phi, & \alpha_4(\beta) &\equiv e^{[\chi]}(\beta) = \phi. \end{aligned} \tag{4.3}$$

These correspond to the simple roots of F_4^{++} whose Dynkin diagram is displayed in table 5. Thus, in the BKL-limit the billiard dynamics is controlled by the Weyl group of F_4^{++} and since this is a hyperbolic Coxeter group, the theory exhibits chaotic behaviour.

¹¹We have corrected some minor misprints in [13].

We now analyze the billiard structure in the low-energy effective theory resulting from compactification on an arbitrary five-dimensional manifold \mathcal{M}_5 . Since we only have 1- and 2-forms in the Lagrangian, only the first, second and third Betti numbers will affect the zero-mode spectrum after compactification.

4.1.1 The $b_1(\mathcal{M}_5) = 0$ compactification

Compactification on a manifold \mathcal{M}_5 with vanishing first Betti number $b_1(\mathcal{M}_5) = 0$, projects out the dominant electric wall $e_1^{[A-]}(\beta)$. This pushes the electric wall $e_{12}^{[B]}(\beta)$ to become dominant, and hence the third simple root $\alpha_3(\beta)$ is replaced by

$$\tilde{\alpha}_3(\beta) = \beta^1 + \beta^2 - \phi. \tag{4.4}$$

This root is long, $(\tilde{\alpha}_3|\tilde{\alpha}_3) = 2$, and has non-vanishing negative scalar products with $\alpha_1(\beta)$ and $\alpha_4(\beta)$,

$$(\tilde{\alpha}_3|\alpha_1) = (\tilde{\alpha}_3|\alpha_4) = -1. \tag{4.5}$$

Hence, one can interpret it as a simple root of a new algebra. Since the fourth simple root is short, $(\alpha_4|\alpha_4) = 1$, the resulting algebra is not simply laced, with the asymmetric part of the Cartan matrix given by

$$A_{34} = \frac{2(\tilde{\alpha}_3|\alpha_4)}{(\tilde{\alpha}_3|\tilde{\alpha}_3)} = -1, \quad A_{43} = \frac{2(\alpha_4|\tilde{\alpha}_3)}{(\alpha_4|\alpha_4)} = -2. \tag{4.6}$$

The new set of simple roots gives rise to the Dynkin diagram of the hyperbolic Kac-Moody algebra B_4^{++} , displayed in table 5. We may therefore conclude that the dynamics in the BKL-limit (in the regime of intermediate asymptotic) remains chaotic after compactification. The simple roots $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4\}$ all correspond to real positive roots of F_4^{++} and therefore this indeed corresponds to a regular embedding of B_4^{++} into F_4^{++} ,

$$\bar{\mathfrak{g}}[b_1] = B_4^{++} \subset F_4^{++}. \tag{4.7}$$

This example thus exhibits a “jump” between two oxidation chains. After compactification, the billiard structure is that of B_4^{++} , which usually is associated with the billiard of a six-dimensional theory with p -form spectrum given by a dilaton, a Maxwell field and a 2-form. These two, seemingly different, theories exhibit identical behaviour in the BKL-limit if the F_4 -theory is compactified on \mathcal{M}_5 with $b_1(\mathcal{M}_5) = 0$. It is tempting to speculate that the preserved algebraic structure reflects that this particular reduction is a consistent truncation of the original theory.

4.1.2 The $b_1(\mathcal{M}_5) = b_2(\mathcal{M}_5) = 0$ compactification

We proceed with the analysis of the F_4 -sequence to reduction on a manifold with $b_1(\mathcal{M}_5) = b_2(\mathcal{M}_5) = 0$. The original set of dominant walls is unchanged by taking \mathcal{M}_5 with $b_2(\mathcal{M}_5) = 0, b_3(\mathcal{M}_5) = 0$ or $b_2(\mathcal{M}_5) = b_3(\mathcal{M}_5) = 0$, so these constraints have no influence on the chaotic behaviour. However, the combination of $b_1(\mathcal{M}_5) = 0$ and $b_2(\mathcal{M}_5) = 0$ drastically changes the set of non-gravitational walls of the theory and so is sufficient to remove chaos.

As we saw in the first case above, the constraint $b_1(\mathcal{M}_5) = 0$ removes the dominant wall $e_1^{[A^-]}(\beta)$ and replaces it with the electric wall $e_{12}^{[B]}(\beta)$, resulting in the hyperbolic algebra B_4^{++} . Now also the electric wall $e_{12}^{[B]}(\beta)$ is removed by the additional constraint $b_2(\mathcal{M}_5) = 0$. Actually, we also kill the magnetic walls of the axion χ and of the one-forms A^\pm so it turns out that the new dominant wall is the gravity wall $G_{145}(\beta)$, i.e.,

$$\tilde{\alpha}_3(\beta) = 2\beta^1 + \beta^2 + \beta^3. \tag{4.8}$$

This positive root can be interpreted as a simple root of a new algebra, and connects with a single link to α_0 and α_2 , resulting in the Dynkin diagram of A_3^{++} which is hyperbolic. However, we also have the axion electric wall $e^{[\chi]}(\phi)$ which has vanishing scalar product with all the other simple roots and hence represents a disconnected A_1 -factor. The resulting algebra is therefore the direct sum

$$\bar{\mathfrak{g}}[b_1, b_2] = A_3^{++} \oplus A_1 \subset F_4^{++}. \tag{4.9}$$

Because of **Fact 3** (section 3.4) this theory is non-chaotic.

4.2 The E_7 -sequence

The compactifications of the E_7 -theory considered here provide a complete counterpart to the examples studied in the previous section since no compactification yields a Coxeter simplex.¹² Here we show explicitly that for these compactifications the dominant walls of the massless p -form spectrum violate the conditions for a valid root system of a Kac-Moody algebra. We study in detail the underlying Coxeter group and determine the structure of the billiard table. We find that the billiard region after compactification is not a fundamental domain, but corresponds to a gallery of finite length inside the Cartan subalgebra of E_7^{++} . We also show that for both compactifications the billiard group \mathfrak{B} corresponds to a non-standard presentation of E_7^{++} .

4.2.1 Dominant wall structure after compactification

The E_7 -sigma model oxidises to a non-supersymmetric consistent truncation of maximal $D = 9$ supergravity with bosonic field content given by a dilaton ϕ , a Maxwell field A and a 3-form C . The electric and magnetic walls are [13]:

$$\begin{aligned} e_i^{[A]}(\beta) &= \beta^i - \frac{2\sqrt{2}}{\sqrt{7}}\phi, & e_{ijk}^{[C]}(\beta) &= \beta^i + \beta^j + \beta^k + \frac{\sqrt{2}}{\sqrt{7}}\phi, \\ m_{i_1, \dots, i_6}^{[A]}(\beta) &= \beta^{i_1} + \dots + \beta^{i_6} + \frac{2\sqrt{2}}{\sqrt{7}}\phi, & m_{ijkl}^{[C]}(\beta) &= \beta^i + \beta^j + \beta^k + \beta^l - \frac{\sqrt{2}}{\sqrt{7}}\phi. \end{aligned} \tag{4.10}$$

¹²Except those compactifications which do not change the dominant walls at all.

The dominant walls are

$$\begin{aligned}
 \alpha_{-1}(\beta) = s_{87}(\beta) &= \beta^8 - \beta^7, & \alpha_0(\beta) = s_{76}(\beta) &= \beta^7 - \beta^6, \\
 & \vdots & & \vdots \\
 \alpha_4(\beta) = s_{32}(\beta) &= \beta^3 - \beta^2, & \alpha_5(\beta) = s_{21}(\beta) &= \beta^2 - \beta^1, \\
 \alpha_6(\beta) = e_1^{[A]}(\beta) &= \beta^1 - \frac{2\sqrt{2}}{\sqrt{7}}\phi, & \alpha_7(\beta) = e_{123}^{[C]}(\beta) &= \beta^1 + \beta^2 + \beta^3 + \frac{\sqrt{2}}{\sqrt{7}}\phi.
 \end{aligned}
 \tag{4.11}$$

The Dynkin diagram constructed from these simple roots coincides with the one of E_7^{++} and is shown in table 2.

The $b_1(\mathcal{M}_8) = 0$ compactification. Taking an eight-dimensional internal manifold \mathcal{M}_8 with vanishing first Betti number removes the dominant electric wall $e_1^{[A]}(\beta)$ and the new dominant wall is the magnetic wall of the 3-form

$$\tilde{\alpha}_6(\beta) = m_{1234}^{[C]}(\beta) = \beta^1 + \beta^2 + \beta^3 + \beta^4 - \frac{\sqrt{2}}{\sqrt{7}}\phi.
 \tag{4.12}$$

However, this positive root is not acceptable as a simple root of a Kac-Moody subalgebra of E_7^{++} since its scalar product with α_7 is positive,

$$(\tilde{\alpha}_6|\alpha_7) = 1.
 \tag{4.13}$$

The billiard table has one obtuse dihedral angle. Although the dominant wall set does not constitute a set of simple roots of a hyperbolic Kac-Moody algebra we may still compute the associated dominant ‘‘coweights’’ (as described in section 3.4), and we find that they are all timelike, revealing that the resulting dynamics is in fact chaotic.

The $b_3(\mathcal{M}_8) = 0$ compactification. The case $b_2(\mathcal{M}_8) = 0$ has no effect on the original wall system so we proceed directly to $b_3(\mathcal{M}_8) = 0$. The dominant electric wall $e_{123}^{[C]}(\beta)$ is projected out which leaves us with three possible candidates for the new dominant wall

$$\begin{aligned}
 m_{123456}^{[A]}(\beta) &= \beta^1 + \dots + \beta^6 + \frac{2\sqrt{2}}{\sqrt{7}}\phi \\
 m_{1234}^{[C]}(\beta) &= \beta^1 + \dots + \beta^4 - \frac{\sqrt{2}}{\sqrt{7}}\phi \\
 G_{178}(\beta) &= 2\beta^1 + \beta^2 + \dots + \beta^6.
 \end{aligned}
 \tag{4.14}$$

The gravity wall G_{178} is subdominant because it can be obtained as the linear combination

$$G_{178}(\beta) = m_{123456}^{[A]}(\beta) + e_1^{[A]}(\beta).
 \tag{4.15}$$

For the remaining two walls it turns out that none can be obtained as an integral linear combination in terms of the other walls. Hence, both walls $m_{123456}^{[A]}$ and $m_{1234}^{[C]}$ are actually dominant. The full set of dominant walls therefore does not define a simplex. Furthermore,

there is one obtuse dihedral angle. One cannot interpret the walls as simple roots of a valid root system.

The resulting structure is best described by choosing $m_{123456}^{[A]}$ as a new simple root $\tilde{\alpha}_7$ and leaving the remaining magnetic wall $m_{1234}^{[C]}$ outside of the algebraic structure. Thus we take

$$\tilde{\alpha}_7(\beta) = \beta^1 + \dots + \beta^6 + \frac{2\sqrt{2}}{\sqrt{7}}\phi. \tag{4.16}$$

This is a long root, $(\tilde{\alpha}_7|\tilde{\alpha}_7) = 2$, and has non-vanishing negative scalar products with α_0 and α_6 ,

$$(\tilde{\alpha}_7|\alpha_0) = -1, \quad (\tilde{\alpha}_7|\alpha_6) = -1, \tag{4.17}$$

implying that the new node in the Dynkin diagram is connected with a single link to nodes 0 and 6. This therefore gives the Dynkin diagram of A_7^{++} which is hyperbolic. Because of the remaining magnetic wall $m_{1234}^{[C]}$ the complete set of dominant walls defines a nine-dimensional bounded region, which is smaller than the fundamental domain of A_7^{++} , and hence has also finite volume: adding the extra wall $m_{1234}^{[C]}$ can never render the volume of the billiard infinite and therefore the dynamics is still chaotic.

The algebra A_7^{++} is usually associated with the billiard of pure gravity in $D = 10$ dimensions, so it is interesting that we here find an embedding of A_7^{++} into E_7^{++} which involves both a magnetic and an electric simple root (although it does not describe the complete dominant wall structure for this theory).

4.2.2 Non-standard presentations of E_7^{++}

For the compactifications studied in the previous section, there is no longer an interpretation of the wall structure in the reduced theory as simple roots of a Kac-Moody algebra. The billiard reflections of the compactified theory, however, still generate the Coxeter group E_7^{++} , although they do not provide a standard presentation.

Let s_i ($i = -1, 0, 1, \dots, 7$) denote the nine fundamental reflections of the Weyl group E_7^{++} .¹³ As explained in section 2, this group is completely determined by its Coxeter exponents m_{ij} , and the defining relations $(s_i s_j)^{m_{ij}} = 1$. This gives the following Coxeter presentation of E_7^{++} :

$$E_7^{++} = \langle s_{-1}, s_0, s_1, \dots, s_7 \mid (s_i s_j)^{m_{ij}} = 1, \quad i, j = -1, 0, 1, \dots, 7 \rangle, \tag{4.18}$$

where the Coxeter exponents m_{ij} follow from the dihedral angles of the fundamental Weyl chamber of E_7^{++} and can be read off from its Coxeter graph/Dynkin diagram, displayed in table 2.

The $b_1(\mathcal{M}_8) = 0$ compactification. Consider now compactification on a manifold \mathcal{M}_8 with $b_1(\mathcal{M}_8) = 0$. After compactification, the dominant set of wall forms is $\{\alpha_{-1}, \alpha_0, \alpha_1, \dots, \tilde{\alpha}_6, \alpha_7\}$. The new dominant walls give rise to a set of reflections

$$\mathbb{S} = \{s_{-1}, s_0, s_1, \dots, \tilde{s}_6, s_7\}. \tag{4.19}$$

¹³As is common in the literature we refer to the crystallographic Coxeter group by the same name as for the associated Kac-Moody algebra.

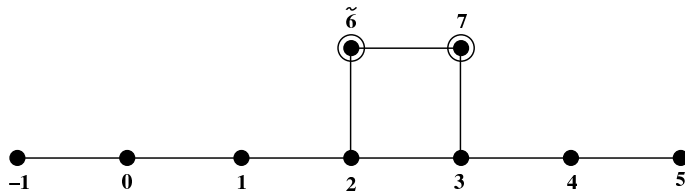


Figure 6: The Coxeter graph of the formal Coxeter group \mathfrak{C} generated by the fundamental reflections $s_{-1}, s_0, s_1, \dots, \tilde{s}_6, s_7$ of the billiard associated with $b_1(\mathcal{M}_8) = 0$. The circles around the nodes $\tilde{6}$ and 7 indicate that the dihedral angle between the associated walls is obtuse. The billiard group \mathfrak{B} is a quotient of \mathfrak{C} and turns out to be E_7^{++} .

We wish to find which is the Coxeter group generated by these reflections, i.e., the corresponding billiard group \mathfrak{B} . The formal Coxeter group \mathfrak{C} associated with these reflections is determined by the Coxeter graph in figure 6 as the quotient group $\mathfrak{C} = \tilde{\mathfrak{C}}/\mathfrak{N}$, where \mathfrak{N} is the normal subgroup generated by the relations among the elements of the freely generated group $\tilde{\mathfrak{C}}$ as encoded in the Coxeter exponents defined by the Coxeter graph.

This is not the end of the story, though, because there are additional relations between the generators, apart from the standard Coxeter relations. Indeed, the new fundamental reflection \tilde{s}_6 can be written in terms of the original reflections as follows

$$\tilde{s}_6 = s_7 s_3 s_4 s_5 s_6 s_5 s_4 s_3 s_7, \quad (4.20)$$

which in turn can be inverted to obtain the old reflection s_6 in terms of the new ones,

$$s_6 = s_5 s_4 s_3 s_7 \tilde{s}_6 s_7 s_3 s_4 s_5. \quad (4.21)$$

This implies that there is one (independent) extra relation among the generators of the billiard group \mathfrak{B}

$$(s_5 s_4 s_3 s_7 \tilde{s}_6 s_7 s_3 s_4 s_5 s_2)^2 = 1. \quad (4.22)$$

This relation is inherited, via Eq. (4.21), from the relation $(s_6 s_2)^2 = 1$ in the original Coxeter group E_7^{++} . Consider the normal subgroup \mathfrak{J} of \mathfrak{C} generated by

$$(s_5 s_4 s_3 s_7 \tilde{s}_6 s_7 s_3 s_4 s_5 s_2)^2. \quad (4.23)$$

Taking the quotient by this subgroup gives a group which is isomorphic to E_7^{++} ,

$$E_7^{++} = \mathfrak{C}/\mathfrak{J} \quad (4.24)$$

since it contains all the fundamental reflections s_i ($i = -1, 0, 1, \dots, 7$) and their relations. This is, however, a non-standard representation of the Coxeter group E_7^{++} .

We see therefore that after compactification the billiard dynamics is still controlled by the hyperbolic Coxeter group E_7^{++} , albeit with a non-standard presentation. This indicates that the Coxeter group structure is a more rigid structure which may survive even when the billiard table is not the original Coxeter polyhedron.

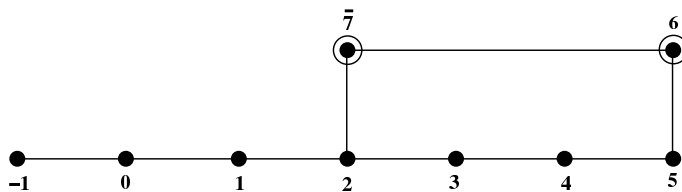


Figure 7: The Coxeter graph of the formal Coxeter group \mathfrak{R} generated by the fundamental reflections $s_{-1}, s_0, s_1, \dots, s_6, \bar{s}_7$ of the billiard associated with $b_3(\mathcal{M}_8) = 0$ (leaving away the magnetic wall $m_{123456}^{[A]}$). The circles around the nodes $\bar{7}$ and 6 indicate that the dihedral angle between the associated walls is obtuse. The billiard group \mathfrak{B} is a quotient of \mathfrak{R} and turns out to be E_7^{++} .

The $b_3(\mathcal{M}_8) = 0$ compactification. We now determine the Coxeter group of the $b_3(\mathcal{M}) = 0$ compactification. It is convenient to initially leave the magnetic wall $m_{123456}^{[A]}$ outside of the analysis. Thus, we take

$$\bar{\alpha}_7 = m_{1234}^{[C]} = \beta^1 + \dots + \beta^4 - \frac{\sqrt{2}}{\sqrt{7}}\phi, \tag{4.25}$$

as the new “simple root”. We denote the Coxeter group freely generated by the set $\mathcal{R} = \{s_{-1}, s_0, s_1, \dots, s_6, \bar{s}_7\}$ by $\tilde{\mathfrak{R}}$, the normal subgroup defined by the Coxeter exponents corresponding to the dihedral angles of the billiard table by \mathfrak{P} , and the corresponding formal Coxeter group by $\mathfrak{R} = \tilde{\mathfrak{R}}/\mathfrak{P}$. The associated Coxeter graph is displayed in figure 7.

Similarly to the previous case we can express the original fundamental reflection s_7 in terms of the new ones as

$$s_7 = s_3 s_4 s_5 s_6 \bar{s}_7 s_6 s_5 s_4 s_3. \tag{4.26}$$

The relation $(s_7 s_2)^2 = 1$ then induces the following relation among the generators of \mathfrak{R} :

$$(s_3 s_4 s_5 s_6 \bar{s}_7 s_6 s_5 s_4 s_3 s_2)^2 = 1, \tag{4.27}$$

which is not a standard Coxeter-type relation. Again, this implies that \mathfrak{R} has a normal subgroup \mathfrak{J} , generated by

$$(s_3 s_4 s_5 s_6 \bar{s}_7 s_6 s_5 s_4 s_3 s_2)^2, \tag{4.28}$$

and hence by taking the quotient by \mathfrak{J} we find yet another non-standard presentation of E_7^{++} ,

$$E_7^{++} = \mathfrak{R}/\mathfrak{J}. \tag{4.29}$$

Adding the reflection in the wall that has been dropped does not modify the result since this reflection can in fact be expressed in terms of the other ones. Thus, this compactification also preserves the Coxeter group E_7^{++} of the billiard, even though it yields a system of walls that cannot be interpreted as the simple roots of E_7^{++} .

4.2.3 The volume of the billiard gallery in the case $b_1(\mathcal{M}_8) = 0$

Here we give an explicit description of the billiard region as a union of images of the fundamental region under the Coxeter group of the noncompact theory, for the compactification

with $b_1(\mathcal{M}_8) = 0$ studied above. We compute the volume of the billiard gallery and show that it is finite, which implies that the reduced theory is chaotic.

As usual, we associate a fundamental reflection $s_i \in E_7^{++}$ to each of the simple roots of E_7^{++} . As we have seen, the compactification with $b_1(\mathcal{M}_8) = 0$ amounts to dropping the wall located at $\alpha_6(\beta) = 0$ and replacing it by $\tilde{\alpha}_6(\beta) = 0$. Thus, one allows for the possibility that the billiard ball moves on the negative side of the electric wall $\alpha_6(\beta) = 0$, i.e., to the region $\alpha_6(\beta) \leq 0$. One can go to that region from the fundamental Weyl chamber \mathcal{F} of E_7^{++} by performing a Weyl reflection in the electric wall which is removed. Starting from the region $\mathcal{B}_1 \equiv \mathcal{F}$, we shall thus apply the reflection s_6 to all the bounding walls of \mathcal{B}_1 . The result is

$$\begin{aligned} s_6(\alpha_6) = -\alpha_6 &\iff \alpha_6(\beta) \leq 0, \\ s_6(\alpha_5) = \alpha_5 + \alpha_6 &\iff (\alpha_5 + \alpha_6)(\beta) \geq 0, \end{aligned} \tag{4.30}$$

with all other walls left invariant by s_6 . In this way we obtain a new region \mathcal{B}_2 , defined by the image of s_6 on \mathcal{F} , i.e.,

$$\begin{aligned} \mathcal{B}_2 &= s_6 \cdot \mathcal{F} \\ &= \{\beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \dots, \alpha_4(\beta) \geq 0, \alpha_7(\beta) \geq 0, (\alpha_5 + \alpha_6)(\beta) \geq 0, \alpha_6(\beta) \leq 0\}. \end{aligned} \tag{4.31}$$

At this point we still have $\mathcal{B}_1 \cup \mathcal{B}_2 \subset \mathcal{B}$ as a proper inclusion, i.e., we have not yet covered all of \mathcal{B} . To proceed we must also allow for the possibility that $(\alpha_5 + \alpha_6)(\beta) \leq 0$, which corresponds to performing a Weyl reflection in the wall $(\alpha_5 + \alpha_6)(\beta) = 0$. A quick calculation reveals that the relevant reflection is the combination $s_5 s_6 s_5 \in E_7^{++}$, as is evident from

$$s_5 s_6 s_5(\alpha_5 + \alpha_6) = -(\alpha_5 + \alpha_6). \tag{4.32}$$

The remaining reflections which are affected by $s_5 s_6 s_5$ are

$$\begin{aligned} s_5 s_6 s_5(\alpha_4) = \alpha_4 + \alpha_5 + \alpha_6 &\iff (\alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0, \\ s_5 s_6 s_5(\alpha_6) = -\alpha_5 &\iff \alpha_5(\beta) \geq 0. \end{aligned} \tag{4.33}$$

We see that there is a new constraint $(\alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0$, and we obtain the region \mathcal{B}_3 , defined as

$$\begin{aligned} \mathcal{B}_3 &= s_5 s_6 s_5 \cdot \mathcal{B}_2 \\ &= \{\beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \dots, \alpha_3(\beta) \geq 0, \alpha_5(\beta) \geq 0, \alpha_7(\beta) \geq 0, \\ &\quad (\alpha_5 + \alpha_6)(\beta) \leq 0, (\alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0\}. \end{aligned} \tag{4.34}$$

The next step is to allow for the possibility $(\alpha_4 + \alpha_5 + \alpha_6)(\beta) \leq 0$, which corresponds to a Weyl reflection with respect to the wall $(\alpha_4 + \alpha_5 + \alpha_6)(\beta) = 0$. The particular combination of fundamental reflections corresponding to this reflection is $s_4 s_5 s_6 s_5 s_4 \in E_7^{++}$,

$$s_4 s_5 s_6 s_5 s_4(\alpha_4 + \alpha_5 + \alpha_6) = -(\alpha_4 + \alpha_5 + \alpha_6). \tag{4.35}$$

The action of $s_4s_5s_6s_5s_4$ on the remaining (non-invariant) walls of \mathcal{B}_3 yields

$$\begin{aligned} s_4s_5s_6s_5s_4(\alpha_3) = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 &\iff (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0, \\ s_4s_5s_6s_5s_4(\alpha_5) = \alpha_5 &\iff \alpha_5(\beta) \geq 0, \\ s_4s_5s_6s_5s_4(\alpha_5 + \alpha_6) = -\alpha_4 &\iff \alpha_4(\beta) \geq 0. \end{aligned} \tag{4.36}$$

As before, we see that a new constraint $(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0$ arises. Putting things together we find that the region \mathcal{B}_4 is given by

$$\begin{aligned} \mathcal{B}_4 &= s_4s_5s_6s_5s_4 \cdot \mathcal{B}_3 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \dots, \alpha_2(\beta) \geq 0, \alpha_4(\beta) \geq 0, \alpha_5(\beta) \geq 0, \alpha_7(\beta) \geq 0, \\ &\quad (\alpha_4 + \alpha_5 + \alpha_6)(\beta) \leq 0, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0 \}. \end{aligned} \tag{4.37}$$

Following the same route, we now allow for $(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \leq 0$. The associated Weyl reflection is $\bar{s} \equiv s_3s_4s_5s_6s_5s_4s_3$, which follows from

$$\bar{s}(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) = -(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6). \tag{4.38}$$

The action of \bar{s} on \mathcal{B}_4 then yields

$$\begin{aligned} \bar{s}(\alpha_2) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 &\iff (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0, \\ \bar{s}(\alpha_4) = \alpha_4 &\iff \alpha_4(\beta) \geq 0, \\ \bar{s}(\alpha_5) = \alpha_5 &\iff \alpha_5(\beta) \geq 0, \\ \bar{s}(\alpha_7) = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 &\iff (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0, \\ \bar{s}(\alpha_4 + \alpha_5 + \alpha_6) = -\alpha_3 &\iff \alpha_3(\beta) \geq 0. \end{aligned} \tag{4.39}$$

Here we see that the new constraint $(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0$, replacing $\alpha_7(\beta) \geq 0$, coincides with the constraint $\tilde{\alpha}_6(\beta) \geq 0$, associated with the new ‘‘simple root’’ $\tilde{\alpha}_6 = m_{1234}^{[C]}(\beta)$, and we may thus conclude that we have reached the far end of the region \mathcal{B} . However, we also uncover a new wall inequality $(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0$, arising because of the ‘‘vertex’’ adjoining nodes 2, 4 and 7 to node 3 in the Dynkin diagram of E_7^{++} (see table 2). The region \mathcal{B}_5 is thus given by

$$\begin{aligned} \mathcal{B}_5 &= \bar{s} \cdot \mathcal{B}_4 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \alpha_0(\beta) \geq 0, \alpha_1(\beta) \geq 0, \alpha_3(\beta) \geq 0, \alpha_4(\beta) \geq 0, \\ &\quad \alpha_5(\beta) \geq 0, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \leq 0, (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0, \\ &\quad (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0 \}. \end{aligned} \tag{4.40}$$

Although we have reached one edge of the region \mathcal{B} , we have not yet exhausted it. To proceed, we must reflect in the wall $(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) = 0$, for which the desired

reflection is $s_2s_3s_4s_5s_6s_5s_4s_3s_2$. We shall not display any more details of the calculations but merely state the result. We obtain a new region \mathcal{B}_6 , given by

$$\begin{aligned} \mathcal{B}_6 &= s_2s_3s_4s_5s_6s_5s_4s_3s_2 \cdot \mathcal{B}_5 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \alpha_0(\beta) \geq 0, \alpha_2(\beta) \geq 0, \dots, \alpha_5(\beta) \geq 0, \\ &\quad (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \leq 0, (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) \geq 0, \\ &\quad (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0 \}. \end{aligned} \tag{4.41}$$

It is satisfactory to see that the far edge $(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0$ of the region \mathcal{B} is preserved. We proceed to reflect in $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\beta) = 0$, which yields

$$\begin{aligned} \mathcal{B}_7 &= s_1s_2s_3s_4s_5s_6s_5s_4s_3s_2s_1 \cdot \mathcal{B}_6 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \alpha_1(\beta) \geq 0, \dots, \alpha_5(\beta) \geq 0, (\alpha_1 + \dots + \alpha_6)(\beta) \leq 0, \\ &\quad (\alpha_0 + \dots + \alpha_6)(\beta) \geq 0, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0 \}. \end{aligned} \tag{4.42}$$

Proceeding in this fashion we find the two additional regions

$$\begin{aligned} \mathcal{B}_8 &= s_0s_1s_2s_3s_4s_5s_6s_5s_4s_3s_2s_1s_0 \cdot \mathcal{B}_7 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_0(\beta) \geq 0, \dots, \alpha_5(\beta) \geq 0, (\alpha_0 + \dots + \alpha_6)(\beta) \leq 0, \\ &\quad (\alpha_{-1} + \dots + \alpha_6)(\beta) \geq 0, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0 \}, \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} \mathcal{B}_9 &= s_{-1}s_0s_1s_2s_3s_4s_5s_6s_5s_4s_3s_2s_1s_0s_{-1} \cdot \mathcal{B}_8 \\ &= \{ \beta \in \mathfrak{h} \mid \alpha_{-1}(\beta) \geq 0, \dots, \alpha_5(\beta) \geq 0, (\alpha_{-1} + \dots + \alpha_6)(\beta) \leq 0, \\ &\quad (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\beta) \geq 0 \}. \end{aligned} \tag{4.44}$$

We see now that there is no longer any remaining wall to reflect in without ending up outside of \mathcal{B} . Thereby we have achieved our goal in covering all of \mathcal{B} by a finite sequence of chambers, and we have the union

$$\mathcal{B} = \bigcup_{i=1}^9 \mathcal{B}_i. \tag{4.45}$$

Since any two consecutive chambers \mathcal{B}_i and \mathcal{B}_{i+1} are adjacent, the new billiard region is naturally described by a gallery Γ connecting \mathcal{B}_1 with \mathcal{B}_9 ,

$$\Gamma : \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_8, \mathcal{B}_9. \tag{4.46}$$

The length of Γ is $k = 9$ so the volume of \mathcal{B} is $9 \cdot \text{vol } \mathcal{F}$, which is finite.

5. Concluding remarks

We have presented the complete compactification analysis for all of the Einstein-dilaton- p -form systems associated with split real forms of finite-dimensional Lie algebras. These are the maximal oxidations of models whose U-duality groups in three spacetime dimensions are finite simple Lie groups. We have extended previous studies of the billiards associated with these models by including the effects of compactification on manifolds of nontrivial topology, which influences the billiard system (relevant to the pre-Planckian regime) through the zero-mode spectrum.

In most cases the algebraic interpretation of the dominant wall system as simple roots of a Kac-Moody algebra is unavailable, but in all cases the Coxeter group structure is preserved. This appears to indicate that the discrete symmetry groups revealed in the BKL-limit are more rigid than the continuous Kac-Moody symmetries. It is possible that this fact is intimately related to the underlying quantum U-duality groups $\mathcal{U}(\mathbb{Z})$ which indeed contain Coxeter groups as subgroups.

On the mathematical side we have found that compactification reveals interesting new structures of the cosmological billiard. In particular, we showed that the geometric reflections with respect to the dominant walls after compactification generate a Coxeter group which generically is described by a non-standard presentation, this being closely linked to the fact that the new billiard region is not a fundamental domain of the reflection group. Using technology from the theory of buildings we also found that this billiard region has a natural description as a gallery, i.e., an ordered sequence of Weyl chambers inside the Cartan subalgebra of the overextended U-duality algebra \mathfrak{u}^{++} , whose Weyl group controlled the billiard dynamics before compactification.

One of the motivations for this work was to determine whether compactification on topologically nontrivial manifolds may eliminate chaos, which would be favorable for big crunch/big bang models which rely on smooth data during the collapse towards the singularity [49–52]. Our results indicate that this can indeed be achieved for some compactifications. For the E_8 -sequence, which is relevant for M-theory and IIA/IIB string theory, chaos can be removed by taking an internal manifold \mathcal{M} with $b_3(\mathcal{M}) = b_4(\mathcal{M}) = 0$. This seems to imply that the simplest Calabi-Yau or G_2 -holonomy manifolds are unsuitable for these constructions. For the heterotic string, whose billiard is described by B_8^{++} , chaos is removed for cases where all spatial dimensions are compact, with the exception of those for which only the first, or only the second, Betti number of the internal manifold vanishes. More generally, for models with no dilaton in dimensions ≤ 10 , chaos cannot be removed by the methods described here since the gravitational wall always remains and pure gravity is known to be chaotic [5]. However, for models with no dilaton in dimension > 10 , or for models with dilatons, chaos can certainly be removed for example by dropping all the wall forms since the resulting models are non-chaotic [5, 6, 68, 36, 38] (the axion wall that remains in the case of F_4 does not invalidate the conclusion). We anticipate that these results will help to serve as a “selection principle” for future investigations of big crunch/big bang transitions in a compact setting.

Although we have focussed on theories exhibiting split U-duality symmetries in three

dimensions, our analysis also applies to the case of non-split real forms. This is due to the fact that when the U-duality algebra \mathfrak{u} is non-split, and the restricted root system of \mathfrak{u} is reduced, the billiard dynamics is controlled by the overextension \mathfrak{f}^{++} of the maximal split “subalgebra” \mathfrak{f} of \mathfrak{u} [16] (see also [59]). All the algebras \mathfrak{f}^{++} are part of the classification considered here, and hence our results cover also these cases. The analysis can easily be extended also to the few cases where the restricted root system of \mathfrak{u} is non-reduced.

Finally, we would like to mention that our work appears to be closely related to that of [69, 70]. These authors consider an M-theory setup where all spatial dimensions are compactified on certain \mathbb{Z}_n -orbifolds of T^{10} , i.e., where the spacetime manifold is of the form $\mathbb{R} \times T^{10}/\mathbb{Z}_n$. The “moduli space” of M-theory on T^{10} is described by the arithmetic quotient $E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R})/K(E_{10})$, and the orbifold projection reduces the global symmetry to a subgroup $\mathcal{G} \subset E_{10}$. A surprising feature of the analysis of [70] is that for some orbifolds the algebra $\mathfrak{g} = \text{Lie } \mathcal{G}$ is a Borcherds subalgebra of E_{10} . We expect that a similar phenomenon would occur in our analysis by allowing for internal manifolds of the form \mathcal{M}/\mathbb{Z}_n . For example, projecting out gravity walls would expose subdominant walls associated with imaginary roots of E_{10} , which could give rise to billiards controlled by Borcherds algebras. Since the Weyl groups of Borcherds algebras are defined only with respect to the real simple roots, and the walls corresponding to non-real simple roots are lightlike, one might expect that chaos will always be removed for these cases.¹⁴ We leave an investigation of this for future work.

Acknowledgments

We thank Sophie de Buyl, Jarah Evslin, Gary Gibbons, Axel Kleinschmidt, Bengt E. W. Nilsson, Jakob Palmkvist, Malcolm Perry, Christoffer Petersson, Larus Thorlacius and Alexander Wijns for discussions at various stages in the preparation of this article. We are also very grateful to Jakob Palmkvist for carefully reading the manuscript and giving numerous useful suggestions. Finally, D.H.W. and D.P. would like to thank, respectively, Université Libre de Bruxelles and the Centre for Theoretical Cosmology at Cambridge University for generous hospitality while this work was carried out.

Work supported in part by IISN-Belgium (convention 4.4511.06 (M.H.) and convention 4.4514.08 (M.H. and D.P)), by the Belgian National Lottery, by the European Commission FP6 RTN programme MRTN-CT-2004-005104 (M.H. and D.P.), and by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11.

A. The complete classification

In this appendix we present the results obtained for all compactifications of the models associated with finite simple Lie groups. The A_n series, representing pure gravity is not affected by the compactifications we consider here and hence is not presented.

Note that the low-rank examples of the infinite families B_n and D_n must be treated separately. This is because for compact orientable N -dimensional compactification mani-

¹⁴We thank Axel Kleinschmidt for discussions on this issue.

folds we have $b_j = b_{N-j}$ and so not all of the Betti numbers are independent. Therefore, we cannot set independently to zero the Betti numbers relevant to the p -forms in the problem when the dimension of space is “too small”. Accordingly, for these infinite families we present the “large- n ” and “small- n ” cases in separate tables.

For each Einstein- p -form system and compactification, we give the Coxeter graph of the formal Coxeter group associated with the billiard. Obtuse angles between the dominant walls are indicated by circles around the corresponding nodes in the Coxeter graph. When the billiard table is a Coxeter simplex we give the Dynkin diagram. Our wall conventions follow those of [13]. The symmetry walls are numbered such that the one corresponding to the overextended node has the highest power of the scale factors, i.e.,

$$\alpha_{-1}(\beta) = \beta^n - \beta^{n-1}, \quad \alpha_0(\beta) = \beta^{n-1} - \beta^{n-2}, \dots \text{etc.} \quad (\text{A.1})$$

The non-symmetry dominant walls are given separately for each case in the forthcoming tables.

The computations in this section were carried out with the assistance of a custom computer program, which is freely available for download at:
<http://www.damtp.cam.ac.uk/user/dhw22/code/>.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\},$ $\{b_1\},$ $\{b_2\},$ $\{b_1, b_2\}$	E_6^{++}	The uncompactified case. The oxidation endpoint is $D = 8$ and comprises a dilaton ϕ , an axion χ and a 3-form C . This theory can be obtained as a truncation of eleven-dimensional supergravity [54]. The non-symmetry dominant walls are the electric walls of C and χ , $\alpha_4(\beta) = \beta^1 + \beta^2 + \beta^3 - \phi/\sqrt{2}$ and $\alpha_5(\beta) = \sqrt{2}\phi$, respectively [13].		Yes
$\{b_3\},$ $\{b_2, b_3\},$	X_9	In this compactification only the electric and magnetic walls of the axion χ are retained. Because the scalar products between 0-form walls and gravity walls are always zero (see section 3.1), the axion walls and the gravity wall can peacefully coexist in this case while still giving rise to a Coxeter polyhedron, which is not, however, a simplex. Its Gram matrix is degenerate. The Coxeter group X_9 is therefore a non-simplex Coxeter group in 7-dimensional hyperbolic space (9 faces). The matrix \bar{A} is a valid Cartan matrix. However, this Cartan matrix is degenerate, and hence the associated Kac-Moody algebra $\hat{\mathfrak{g}}(\bar{A})$ is not simple [17]. In fact, it contains a one-dimensional ideal $\mathfrak{i} = \mathbb{R}k$, where $k \in \mathfrak{h} \subset E_6^{++}$ is the linear combination of Cartan generators of E_6^{++} with coefficients given by the components of the null vector v which spans the kernel of \bar{A} . See [67] for a detailed discussion of this phenomenon.		Yes
$\{b_1, b_3\},$ $\{b_1, b_2,$ $b_3\}$	$A_5^{++} \times$ A_1	Here, only the electric wall of the axion is retained, which has zero scalar product with the gravity and symmetry walls. Hence, the Coxeter group corresponds to the direct product of the "pure gravity group" A_5^{++} and the group A_1 generated by reflections in the electric wall. The volume of the billiard region is infinite.		No

Table 1: The E_6 -sequence. The leftmost column indicates which Betti numbers vanish.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\},$ $\{b_2\},$ $\{b_1, b_2\}$	E_7^{++}	The uncompactified case. This theory oxidises to $D = 9$ where it corresponds to a consistent non-supersymmetric truncation of maximal $D = 9$ supergravity. The field content includes a dilaton ϕ , and Maxwell field A and a 3-form C [54]. The non-symmetry dominant walls are the electric walls of A and C , $\alpha_6(\beta) = \beta^1 - \frac{2\sqrt{2}}{\sqrt{7}}\phi$ and $\alpha_7(\beta) = \beta^1 + \beta^2 + \beta^3 + \frac{\sqrt{2}}{\sqrt{7}}\phi$, respectively [13].		Yes
$\{b_1\}$	E_7^{++}	Compactification on a manifold \mathcal{M} with $b_1 = 0$ projects out the dominant wall α_6 and promotes the magnetic wall form of the 3-form, $\tilde{\alpha}_6 = m_{1234}^{[C]}$, to become dominant. The resulting set of wall forms does not define an acute-angled polyhedron. However, we may still associate a Coxeter graph to this wall system, which defines its formal Coxeter group \mathfrak{C} . The group \mathfrak{C} has a normal subgroup \mathfrak{J} , and taking the quotient by this subgroup yields the billiard Coxeter group \mathfrak{B} , which turns out to be the original Coxeter group, $E_7^{++} = \mathfrak{C}/\mathfrak{J}$. Therefore the resulting dynamics is chaotic, even though the billiard region does not correspond to the fundamental Weyl chamber of E_7^{++} . See section 4.2 for a detailed discussion of this compactification.		Yes
$\{b_3\}$	E_7^{++}	In this compactification the wall corresponding to the simple root α_7 is projected out. There are two new dominant wall forms. One of them is again the magnetic wall form of the 3-form, $\tilde{\alpha}_7 = m_{1234}^{[C]}$. The other is the wall form $\tilde{\alpha}_7 = m_{123456}^{[A]}$. The billiard region is a finite union of images of the fundamental region of E_7^{++} and hence the dynamics is chaotic. The formal Coxeter group \mathfrak{A} associated with the (non acute-angled) polyhedron defined by all dominant walls except $\tilde{\alpha}_7$ has a normal subgroup \mathfrak{J} , and the quotient $\mathfrak{A}/\mathfrak{J}$ is again equal to E_7^{++} . See section 4.2 for a detailed discussion of this compactification.		Yes
$\{b_1, b_3\}$	D_6^{+++}	This compactification is slightly subtle. Both the magnetic walls $m_{1234}^{[C]}$ and $m_{123456}^{[A]}$ are now dominant, together with the gravity wall G_{178} . Thus the polyhedron is not a simplex and furthermore, the gravity walls intersect the magnetic walls at obtuse angles. The Coxeter graph displayed corresponds to the wall system obtained by excluding the gravity wall. The Coxeter group D_6^{+++} associated with this graph is not hyperbolic (although it is of course Lorentzian). However, we have explicitly checked that the new billiard region does not contain any spacelike rays, and hence the dynamics is nevertheless chaotic. The reason is that the gravity wall comes into play and “cuts” the fundamental region of Y_9 in such a way that it shields the spacelike direction of escape and thus renders the total volume finite. It is noteworthy that a triple-extended Kac-Moody algebra plays a role in the billiard structure.		Yes

Table 2: The E_7 -sequence.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{b_2, b_3\}$	E_7^{++}	The compactification with $b_2 = b_3 = 0$ gives rise to the same Coxeter group structure as the $b_3 = 0$ case. The additional constraint $b_2 = 0$ removes the magnetic wall $m_{123456}^{[A]}$, while preserving the magnetic wall $m_{1234}^{[C]}$ and promoting the gravity wall G_{178} to a dominant wall. Thus, excluding the gravity wall, we find the formal Coxeter group \mathfrak{A} . The quotient group $\mathfrak{A}/\mathfrak{I}$ is again E_7^{++} , and the resulting dynamics is chaotic.		Yes
$\{b_1, b_2, b_3, b_4\}$	A_6^{++}	In this compactification, all p -form walls are projected out. In the absence of the dilaton (pure gravity), we would obtain a chaotic system. However, because the oxidation endpoint of the E_7 -sequence includes a dilaton, which is never projected out, the resulting dominant wall system is not sufficient to provide a finite volume billiard. This can be seen from the fact that there are eight dominant walls, corresponding to the eight simple roots of A_6^{++} , while the total space in which the dynamics takes place is nine-dimensional. Hence, even though A_6^{++} is hyperbolic, the billiard domain is of infinite volume since there is always an extra “dimension” in which the particle may escape to infinity. The compactification $b_1 = b_2 = b_3$ gives rise to the same dynamics since the magnetic wall which is preserved is not sufficient to provide a finite volume billiard.		No

Table 3: The E_7 -sequence continued.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}$	E_8^{++}	The uncompactified case. The oxidation endpoint is eleven-dimensional supergravity, containing a 3-form potential C [54]. The only non-symmetry dominant wall is the electric wall of C , $\alpha_8(\beta) = \beta^1 + \beta^2 + \beta^3$ [13].		Yes
$\{b_3\}$	E_7^{+++}	This compactification renders the magnetic wall of the 3-form dominant, while the electric wall is projected out. However, also the gravity wall becomes dominant, implying that the billiard table is not a simplex, and has obtuse angles. Excluding the gravity wall, the reflections in the new dominant walls give rise to the (non-hyperbolic) Lorentzian Coxeter group E_7^{+++} . The fundamental Weyl chamber of E_7^{+++} is of infinite volume, which would imply non-chaotic dynamics. However, here the gravity wall again comes in and shields the direction of escape, rendering the total volume finite and preserves the chaotic dynamics. It is noteworthy that a triple-extended Kac-Moody algebra plays again a role in the billiard structure.		Yes
$\{b_3, b_4\}$	A_8^{++}	This corresponds to the standard scenario when all of the p -form walls are projected out. The billiard is the same as for pure gravity in eleven dimensions, which is non-chaotic since the Kac-Moody algebra A_8^{++} is not hyperbolic.		No

Table 4: The E_8 -sequence.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}, \{b_2\}, \{b_3\}, \{b_2, b_3\}$	F_4^{+++}	The uncompactified case. The oxidation endpoint is $D = 6$ and the field content is given by a dilaton ϕ , and axion χ , two Maxwell fields A^\pm , a 2-form B and a 3-form potential C with self-dual field strength G [54]. The non-symmetry dominant walls are the electric walls of A^- and χ , $\alpha_3(\beta) = \beta^1 - \phi/2$ and $\alpha_4(\beta) = \phi$, respectively [13].		Yes
$\{b_1\}, \{b_1, b_3\}$	B_4^{+++}	This compactification renders the one-form A^- massive and hence projects out the associated electric wall $\omega_3(\beta) = \beta^1 - \phi/2$. The new dominant wall is the electric wall of the 2-form B , which reads $\tilde{\omega}_3(\beta) = \beta^1 + \beta^2 - \phi$. The new dominant wall may be identified with the simple roots of the hyperbolic Kac-Moody algebra B_4^{+++} , and the reflections in the faces of these walls generate the Weyl group $\mathfrak{W}[B_4^{+++}]$. The billiard domain \mathcal{B} coincides with the fundamental Weyl chamber \mathcal{F} of $\mathfrak{W}[B_4^{+++}]$ and since this is of finite volume, the dynamics is chaotic. This case is discussed in more detail in section 4.1.		Yes
$\{b_1, b_2\}, \{b_1, b_2, b_3\}$	$A_3^{+++} \times A_1$	In this compactification, all p -form walls are projected out, but the axion wall $\omega_4 = \phi$ is always present. Therefore the resulting dominant wall system is the standard one A_3^{+++} of pure gravity in 6 dimensions, augmented with the extra axion wall. Even though A_3^{+++} is hyperbolic, there exists a spacelike coweight due to the finite A_1 -factor, and therefore the dynamics is non-chaotic. This case is discussed in more detail in section 4.1.		No

Table 5: The F_4 -sequence.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}, \{b_2\}, \{b_3\}$	G_2^{+++}	The uncompactified case. The oxidation endpoint is 5-dimensional Maxwell-Einstein gravity [54]. The connection between this supergravity theory and G_2^{+++} has also been extensively investigated in [71]. The only non-symmetry dominant wall is the electric wall of the Maxwell field A , $\alpha_2(\beta) = \beta^1$ [13].		Yes
$\{b_1\}$	A_2^{+++}	This compactification simply renders the one-form A massive and thus projects out the electric wall $\omega_2(\beta) = \beta^1$. The new dominant wall is the gravity wall $\tilde{\omega}_2(\beta) = G_{134}(\beta) = 2\beta^1 + \beta^2$, and not the magnetic wall $m_{123}^{[A]}(\beta) = \beta^1 + \beta^2 + \beta^3$, since we have $m_{123}^{[A]}(\beta) = G_{134}(\beta) + \omega_0(\beta) + \omega_1(\beta)$ (see [13] for our wall conventions). The billiard dynamics is controlled by the Weyl group of the hyperbolic Kac-Moody algebra A_2^{+++} , and is therefore chaotic.		Yes

Table 6: The G_2 -sequence.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}, \{b_2\}$	B_n^{++}	The uncompactified case. The oxidation endpoint is in $D = n + 2$ dimensions and includes a dilaton ϕ , a 2-form B and a Maxwell field A [54]. The non-symmetry dominant walls are the electric wall of the Maxwell field A and the magnetic wall of the 2-form B , $\alpha_n(\beta) = e_1^{[A]}(\beta) = \beta^1 + \frac{a}{2\sqrt{2}}\phi$ and $\alpha_1(\beta) = m_{1\dots(n-2)}(\beta) = \beta^1 + \dots + \beta^{n-2} - \frac{a}{\sqrt{2}}\phi$ ($a^2 = 8/n$) [13].		Yes for $n \leq 8$, no for $n > 8$.
$\{b_1\}$	D_n^{++}	This compactification projects our the electric wall $e_1^{[A]}(\beta)$ corresponding to node n in the original diagram. The new dominant wall is the electric wall of the 2-form $e_{12}^{[B]}(\beta)$ which attaches by a single link to node $n - 2$. The resulting Kac-Moody algebra is D_n^{++} which is hyperbolic for $1 \leq n \leq 8$.		Yes for $n \leq 8$, no for $n > 8$.
$\{b_3\}$	R_{n+2}	Here we project out the magnetic wall $m_{12\dots(n-2)}(\beta)$ of the 2-form, corresponding to node number 1 in the original diagram. The new dominant wall is the magnetic wall $m_{12\dots(n-1)}(\beta)$ which attaches by a double link to node 0. The matrix of scalar products between the dominant walls is a valid Cartan matrix corresponding to the rank $n + 2$ Lorentzian Kac-Moody algebras R_{n+2} . To our knowledge, this class of algebras is not part of any previous classification. It is an extension of the twisted affine algebra $D_{n+1}^{(2)}$ and fulfills the property of [72]. Since the algebras are not hyperbolic, the dynamics is always non-chaotic.		No
$\{b_1, b_2\}$	Z_{n+2}	The electric wall $e_1^{[A]}(\beta)$ of the Maxwell field and the electric wall $e_{12}^{[B]}(\beta)$ of the 2-form are both projected out. As a result the magnetic wall $m_{12\dots(n-2)}(\beta)$ and the gravity wall $G_{1n(n+1)}(\beta)$ become dominant. Because of the facts explained in section 3.1 the resulting billiard table is not acute-angled (but is a simplex). We have displayed the Coxeter graph of the formal Coxeter group associated to the billiard table. It is a (non-hyperbolic) Lorentzian Coxeter group, which we call Z_{n+2} . The billiard group is a quotient of the formal Coxeter group by the normal subgroup generated by the extra relations, which we have not worked out explicitly. We have checked that the volume of the billiard region is infinite so that the dynamics is not chaotic.		No

Table 7: The B_n -sequence for $n > 4$.

$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{b_1, b_3\}$	S_{n+2}	This compactification displays an example of a “coexistence” of an electric, a magnetic and a gravity wall in the set of dominant walls. These correspond to $e_{12}^{[B]}(\beta)$, $m_{12\dots(n-1)}^{[A]}(\beta)$ and $G_{1n(n+1)}(\beta)$. The billiard table is therefore not a simplex. To analyze the situation we may consider the set of dominant walls with the gravity wall excluded. Then the matrix of scalar products between these walls gives rise to the Cartan matrix of a rank $n + 2$ Lorentzian Kac-Moody algebra S_{n+2} , whose Dynkin diagram is displayed to the right. This is a non-standard extension of the affine algebra B_n^+ . The fundamental domain for this algebra has infinite volume, and we have checked that including the gravity wall is not enough to render the billiard region finite. Thus, the dynamics is non-chaotic.		No
$\{b_2, b_3\}$	Q_{n+2}	In this compactification the magnetic wall $\alpha_1(\beta) = m_{1\dots(n-2)}^{[B]}(\beta)$ of the 2-form B is projected out and is replaced by the gravity wall $G_{1n(n+1)}(\beta)$. The number of dominant walls is therefore $n + 2$ and the billiard table is a simplex. However, due to the presence of the gravity wall it is not acute-angled. We display the Coxeter graph of the associated formal Coxeter group, denoted by Q_{n+2} . The region bounded by the new dominant walls contains spacelike coweights and hence the dynamics is non-chaotic.		No
$\{b_1, b_2, b_3\}$	A_{n-1}^{++}	This compactification projects out all p -form walls, and so only the gravity and symmetry walls remain, giving rise to the rank $n-1$ Kac-Moody algebra A_{n-1}^{++} which is hyperbolic for $2 \leq n \leq 8$. Because of the presence of a dilaton, the total space \mathfrak{M}_β is however $n + 2$ -dimensional and hence there is always a direction of escape, rendering the dynamics non-chaotic.		No

Table 8: The B_n -sequence for $n > 4$.

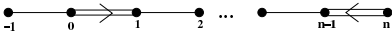
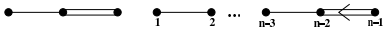
$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}$	C_n^{++}	The uncompactified case. The oxidation endpoint corresponds is gravity in $D = 4$ coupled to a collection of dilatonic scalars, axions and Maxwell fields [54]. See [13] for the definition of the dominant walls in this case.		Yes for $n \leq 4$, no for $n > 4$
$\{b_1 = b_2\}$	$A_1^{++} \times C_{n-1}$	Because of Poincaré duality we have $b_1 = b_2$ so the only relevant compactification is the one for which $b_1 = b_2 = 0$. This renders all Maxwell fields massive, and the resulting theory is described by two disconnected diagrams: the pure gravity piece A_1^{++} , and a piece C_{n-1} associated with the axions. Because of the disconnected part corresponding to a finite Lie algebra we know that there exist spacelike coweights, and hence the dynamics is non-chaotic.		No for $n \geq 2$

Table 9: The C_n -sequence.

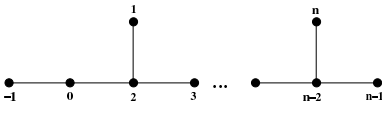
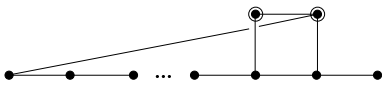
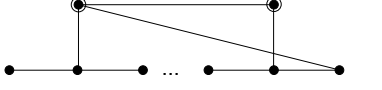
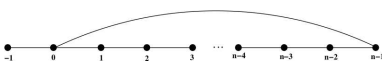
$\{b_i(\mathcal{M}) = 0\}$	Coxeter Group	Comments	Coxeter Graph of formal Coxeter group / Dynkin Diagram	Chaotic?
$\{\}, \{b_1\}$	D_n^{++}	The uncompactified case. The oxidation endpoint is $D = n + 2$, with matter content given by a dilaton ϕ and a 2-form B . The associated electric and magnetic walls, $e_{12}^{[B]}(\beta)$ and $m_{12\dots(n-2)}^{[B]}(\beta)$, are both dominant before compactification [54]. The non-symmetry dominant walls are the electric and magnetic wall of the 2-form B , $\alpha_n(\beta) = \beta^1 + \beta^2 + \frac{a}{\sqrt{2}}\phi$ and $\alpha_1(\beta) = \beta^1 + \dots + \beta^{n-2} - \frac{a}{\sqrt{2}}\phi$, respectively ($a^2 = 8/n$) [13].		Yes for $n \leq 8$, no for $n > 8$.
$\{b_2\}, \{b_1, b_2\}$	Z_{n+2}	Compactification on a manifold \mathcal{M} with $b_2 = 0$ (or $b_1 = b_2 = 0$) renders the electric field massive, and the electric wall is replaced by the gravity wall $G_{1n(n+1)}(\beta)$. This new set of dominant walls do not define an acute-angled billiard table. The formal Coxeter group associated with the reflections in these walls is again the Lorentzian Coxeter group Z_{n+2} , whose Coxeter graph is displayed to the right. This is the same Coxeter group which appears in the billiard of the $b_1 = b_2 = 0$ compactification of the B_n -theory. The billiard region contains spacelike coweights and hence the dynamics is non-chaotic.		No
$\{b_3\}, \{b_1, b_3\}$	W_{n+2}	In this compactification the magnetic field becomes massive and hence the magnetic wall $m_{12\dots(n-2)}^{[B]}(\beta)$ is projected out, again rendering the gravity wall $G_{1n(n+1)}(\beta)$ dominant. The new set of dominant walls does not define an acute-angled billiard table. The Coxeter graph derived from the dominant walls is displayed to the right and it corresponds to a (non-hyperbolic) Lorentzian Coxeter group. We have checked that the fundamental chamber contains spacelike coweights and hence the dynamics is non-chaotic.		No
$\{b_2, b_3\}, \{b_1, b_2, b_3\}$	A_{n-1}^{++}	All p -form walls are projected out and the dominant wall system is the standard one A_{n-1}^{++} for pure gravity. However, since there is a dilaton the total scale-factor space \mathfrak{M}_β is $n + 2$ -dimensional, and there will always exist a direction of escape, rendering the theory non-chaotic.		No

Table 10: The D_n -sequence for $n > 4$.

$\{b_i(\mathcal{M}) = 0\}$	$n = 2$	$n = 3$	$n = 4$
$\{\}, \{b_2\}$	$B_2^{++} \textcircled{C}$	$B_3^{++} \textcircled{C}$	$B_4^{++} \textcircled{C}$
$\{b_1\}$	$D_2^{++} \textcircled{C}$	$D_3^{++} \textcircled{C}$	$D_4^{++} \textcircled{C}$
$\{b_1, b_2\}$	$A_1^{++} \times A_1$	$A_3^{++} \textcircled{C}$	A_3^{++}
$\{b_3\}$	$K_4 \textcircled{C}$	$A_3^{++} \textcircled{C}$	L_6
$\{b_1, b_3\}$	A_1^{++}	$A_3^{++} \textcircled{C}$	A_3^{++}
$\{b_2, b_3\}$	A_1^{++}	$A_3^{++} \textcircled{C}$	L_6
$\{b_1, b_2, b_3\}$	A_1^{++}	$A_3^{++} \textcircled{C}$	A_3^{++}

Table 11: Compactifications of the B_n^{++} -models for small n . Chaotic cases are denoted with \textcircled{C} . The billiard group of the $b_3 = 0$ compactification of the B_2^{++} -theory is the Weyl group of the hyperbolic Kac-Moody algebra K_4 , whose Dynkin diagram is displayed in table 13 (this algebra is denoted by “4-4” in [73]). Since K_4 is hyperbolic, the dynamics is chaotic. The billiard group $\mathfrak{B} := L_6$ which controls the $b_3 = 0$ compactification of the B_4^{++} theory is a quotient of the formal Coxeter group $\mathfrak{C} := L'_6$, whose Coxeter graph is displayed in table 13. The billiard region is not acute-angled and hence is not a fundamental domain of L_6 . Moreover, there exist spacelike coweights and therefore the dynamics is non-chaotic. For all the non-chaotic compactifications in the table which are described by well known hyperbolic Weyl groups, the β -space \mathfrak{M}_β contains extra dilatonic directions which render the volume of the billiard region infinite.

$\{b_i(\mathcal{M}) = 0\}$	$n = 2$	$n = 3$	$n = 4$
$\{\}, \{b_1\}$	$A_3 \times A_1^+$	$A_3^{+++} \textcircled{C}$	$D_4^{++} \textcircled{C}$
$\{b_2\}$	$A_1^{++} \times A_1$	M_5	A_3^{++}
$\{b_1, b_2\}, \{b_2, b_3\}$	$A_1^{++} \times A_1$	A_2^{++}	A_3^{++}
$\{b_1, b_2, b_3\}$	$A_1^{++} \times A_1$	A_2^{++}	A_3^{++}
$\{b_3\}, \{b_1, b_3\}$	A_1^{+++}	N_5	A_3^{++}

Table 12: Compactifications of the D_n^{++} -models for small n . Chaotic cases are denoted with \textcircled{C} . The $b_2 = 0$ compactification of the D_3^{++} theory is governed by the billiard group $\mathfrak{B} := M_5$ which is a quotient of the formal Coxeter group $\mathfrak{C} := M'_5$, as defined through the Coxeter graph in table 13. The billiard region contains obtuse angles and so is not a fundamental domain of M_5 . Furthermore, the billiard region contains spacelike coweights and hence the dynamics is non-chaotic. The story for the billiard group N_5 is similar, and the Coxeter graph associated with the formal Coxeter group N'_5 is given in table 13. The $b_1 = 0$ compactification of the D_3^{++} -theory is controlled by the Weyl group of the Lorentzian Kac-Moody algebra A_3^{+++} . Although this algebra is not hyperbolic, the theory is nevertheless chaotic since the number of dominant walls, 6, exceeds the dimension of β -space, $\dim \mathfrak{M}_\beta = 5$, and the billiard region is of finite volume. This is similar to the X_9 case in table 1. For all the non-chaotic compactifications in the table which are described by well known hyperbolic Weyl groups, the β -space \mathfrak{M}_β contains extra dilatonic directions which render the volume of the billiard region infinite.

K_4		L'_6	
M'_5		N'_5	

Table 13: K_4 represents the Dynkin diagram of a hyperbolic Kac-Moody algebra, whose Weyl group controls the chaotic dynamics of the $b_3 = 0$ compactification of the B_2^{++} -theory. The remaining Coxeter graphs represent the formal Coxeter groups L'_6 , M'_5 and N'_5 associated with various compactifications of the B_n^{++} and D_n^{++} theories (see tables 11 and 12). As usual, the extra circles denote which dihedral angles are obtuse. In these graphs, “ G ” denotes the dominant gravity wall, “ EA ” the dominant electric wall of the Maxwell field A , “ EB ” and “ MB ” denote, respectively, the dominant electric and magnetic walls of the 2-form B . See tables 8 and 10 for a summary of the field contents of these theories.

References

- [1] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, *Ann. Phys. (NY)* **19** (1970) 525.
- [2] V.A. Belinsky, I.M. Khalatnikov and E.M. Lifshitz, *A general solution of the Einstein equations with a time singularity*, *Ann. Phys. (NY)* **31** (1982) 639.
- [3] I.M. Khalatnikov and A.Y. Kamenshchik, *Lev Landau and the problem of singularities in cosmology*, [arXiv:0803.2684](https://arxiv.org/abs/0803.2684).
- [4] C.W. Misner, *Mixmaster universe*, *Phys. Rev. Lett.* **22** (1969) 1071.
- [5] J. Demaret, M. Henneaux and P. Spindel, *Nonoscillatory behavior in vacuum Kaluza-Klein cosmologies*, *Phys. Lett.* **B 164** (1985) 27.
- [6] J. Demaret, J.L. Hanquin, M. Henneaux, P. Spindel and A. Taormina, *The fate of the mixmaster behavior in vacuum inhomogeneous Kaluza-Klein cosmological models*, *Phys. Lett.* **B 175** (1986) 129.
- [7] T. Damour and M. Henneaux, *Chaos in superstring cosmology*, *Phys. Rev. Lett.* **85** (2000) 920 [[hep-th/0003139](https://arxiv.org/abs/hep-th/0003139)].
- [8] D.M. Chitre, Ph.D thesis, University of Maryland (1972).
- [9] C.W. Misner, *The Mixmaster cosmological metrics*, [gr-qc/9405068](https://arxiv.org/abs/gr-qc/9405068).

- [10] T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards*, *Class. and Quant. Grav.* **20** (2003) R145 [[hep-th/0212256](#)].
- [11] T. Damour, M. Henneaux, B. Julia and H. Nicolai, *Hyperbolic Kac-Moody algebras and chaos in Kaluza-Klein models*, *Phys. Lett.* **B 509** (2001) 323 [[hep-th/0103094](#)].
- [12] T. Damour and M. Henneaux, *E_{10} , BE_{10} and arithmetical chaos in superstring cosmology*, *Phys. Rev. Lett.* **86** (2001) 4749 [[hep-th/0012172](#)].
- [13] T. Damour, S. de Buyl, M. Henneaux and C. Schomblond, *Einstein billiards and overextensions of finite- dimensional simple Lie algebras*, *JHEP* **08** (2002) 030 [[hep-th/0206125](#)].
- [14] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [[hep-th/9410167](#)].
- [15] N.A. Obers and B. Pioline, *U-duality and M-theory*, *Phys. Rept.* **318** (1999) 113 [[hep-th/9809039](#)].
- [16] M. Henneaux and B. Julia, *Hyperbolic billiards of pure $D = 4$ supergravities*, *JHEP* **05** (2003) 047 [[hep-th/0304233](#)].
- [17] V. Kac, *Infinite dimensional Lie algebras* 3rd Ed., Cambridge University Press, Cambridge U.K. (1990).
- [18] T. Damour, M. Henneaux and H. Nicolai, *E_{10} and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [[hep-th/0207267](#)].
- [19] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, in *Lectures in Applied Mathematics* **21** (1985) 335 [LPTENS 82/22].
- [20] B. Julia, *Group disintegrations*, Invited paper presented at *Nuffield Gravity Workshop*, Cambridge England, Jun 22 - Jul 12 1980 [LPTENS 80/16].
- [21] P. Breitenlohner and D. Maison, *On the Geroch group*, *Annales. Poincaré Phys. Theor.* **46** (1987) 215.
- [22] H. Nicolai, *The integrability of $N = 16$ supergravity*, *Phys. Lett.* **B 194** (1987) 402.
- [23] H. Nicolai and H. Samtleben, *Integrability and canonical structure of $D = 2$, $N = 16$ supergravity*, *Nucl. Phys.* **B 533** (1998) 210 [[hep-th/9804152](#)].
- [24] A. Kleinschmidt and H. Nicolai, *Gradient representations and affine structures in AE_n* , *Class. and Quant. Grav.* **22** (2005) 4457 [[hep-th/0506238](#)].
- [25] F. Englert, L. Houart, A. Kleinschmidt, H. Nicolai and N. Tabti, *An E_9 multiplet of BPS states*, *JHEP* **05** (2007) 065 [[hep-th/0703285](#)].
- [26] H. Lü, M.J. Perry and C.N. Pope, *Infinite-dimensional symmetries of two-dimensional coset models*, [arXiv:0711.0400](#).
- [27] H. Lü, M.J. Perry and C.N. Pope, *Infinite-dimensional symmetries of two-dimensional coset models coupled to gravity*, [arXiv:0712.0615](#).
- [28] H. Nicolai, *$D = 11$ supergravity with local $SO(16)$ invariance*, *Phys. Lett.* **B 187** (1987) 316.
- [29] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [[hep-th/0104081](#)].
- [30] F. Englert, L. Houart, A. Taormina and P.C. West, *The symmetry of M-theories*, *JHEP* **09** (2003) 020 [[hep-th/0304206](#)].

- [31] F. Englert and L. Houart, G^{+++} -invariant formulation of gravity and M-theories: exact BPS solutions, *JHEP* **01** (2004) 002 [[hep-th/0311255](#)].
- [32] E.A. Bergshoeff, J. Gomis, T.A. Nutma and D. Roest, Kac-Moody spectrum of (half-)maximal supergravities, *JHEP* **02** (2008) 069 [[arXiv:0711.2035](#)].
- [33] F. Riccioni and P.C. West, E_{11} -extended spacetime and gauged supergravities, *JHEP* **02** (2008) 039 [[arXiv:0712.1795](#)].
- [34] B.K. Berger, D. Garfinkle, J. Isenberg, V. Moncrief and M. Weaver, The singularity in generic gravitational collapse is spacelike, local and oscillatory, *Mod. Phys. Lett. A* **13** (1998) 1565 [[gr-qc/9805063](#)].
- [35] B.K. Berger, Numerical approaches to space-time singularities, *Living Rev. Rel.* **1** (1998) 7.
- [36] L. Andersson and A.D. Rendall, Quiescent cosmological singularities, *Commun. Math. Phys.* **218** (2001) 479 [[gr-qc/0001047](#)].
- [37] H. Ringstrom, The Bianchi IX attractor, *Annales Henri Poincaré* **2** (2001) 405 [[gr-qc/0006035](#)].
- [38] T. Damour, M. Henneaux, A.D. Rendall and M. Weaver, Kasner-like behaviour for subcritical Einstein-matter systems, *Annales Henri Poincaré* **3** (2002) 1049 [[gr-qc/0202069](#)].
- [39] D. Garfinkle, Numerical simulations of generic singularities, *Phys. Rev. Lett.* **93** (2004) 161101 [[gr-qc/0312117](#)].
- [40] J.M. Heinzle, C. Uggla and N. Röhr, The cosmological billiard attractor, [gr-qc/0702141](#).
- [41] D. Garfinkle, Numerical simulations of general gravitational singularities, *Class. and Quant. Grav.* **24** (2007) S295 [[arXiv:0808.0160](#)].
- [42] T. Damour and S. de Buyl, Describing general cosmological singularities in Iwasawa variables, *Phys. Rev. D* **77** (2008) 043520 [[arXiv:0710.5692](#)].
- [43] D.H. Wesley, P.J. Steinhardt and N. Turok, Controlling chaos through compactification in cosmological models with a collapsing phase, *Phys. Rev. D* **72** (2005) 063513 [[hep-th/0502108](#)].
- [44] D.H. Wesley, Kac-Moody algebras and controlled chaos, *Class. and Quant. Grav.* **24** (2007) F7 [[hep-th/0610322](#)].
- [45] T. Damour and H. Nicolai, Higher order M-theory corrections and the Kac-Moody algebra E_{10} , *Class. and Quant. Grav.* **22** (2005) 2849 [[hep-th/0504153](#)].
- [46] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge U.K. (1984).
- [47] S.S. Gubser, String production at the level of effective field theory, *Phys. Rev. D* **69** (2004) 123507 [[hep-th/0305099](#)].
- [48] A.J. Tolley and D.H. Wesley, String pair production in a time-dependent gravitational field, *Phys. Rev. D* **72** (2005) 124009 [[hep-th/0509151](#)].
- [49] J. Khoury, B.A. Ovrut, P.J. Steinhardt and N. Turok, The ekpyrotic universe: colliding branes and the origin of the hot big bang, *Phys. Rev. D* **64** (2001) 123522 [[hep-th/0103239](#)].
- [50] P.J. Steinhardt and N. Turok, Cosmic evolution in a cyclic universe, *Phys. Rev. D* **65** (2002) 126003 [[hep-th/0111098](#)].

- [51] A. Buonanno, T. Damour and G. Veneziano, *Pre-big bang bubbles from the gravitational instability of generic string vacua*, *Nucl. Phys.* **B 543** (1999) 275 [[hep-th/9806230](#)].
- [52] M. Gasperini and G. Veneziano, *The pre-big bang scenario in string cosmology*, *Phys. Rept.* **373** (2003) 1 [[hep-th/0207130](#)].
- [53] P. Breitenlohner, D. Maison and G.W. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, *Commun. Math. Phys.* **120** (1988) 295.
- [54] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Higher-dimensional origin of $D = 3$ coset symmetries*, [hep-th/9909099](#).
- [55] A. Keurentjes, *The group theory of oxidation*, *Nucl. Phys.* **B 658** (2003) 303 [[hep-th/0210178](#)].
- [56] E.B. Vinberg et al. (eds.), *Geometry II — Spaces of constant curvature*, Springer (1993).
- [57] T. Damour and M. Henneaux, *Oscillatory behaviour in homogeneous string cosmology models*, *Phys. Lett.* **B 488** (2000) 108 [*Erratum ibid.* **B 491** (2000) 377] [[hep-th/0006171](#)].
- [58] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics (1992).
- [59] M. Henneaux, D. Persson and P. Spindel, *Spacelike singularities and hidden symmetries of gravity*, *Living Rev. Rel.* **11** (2008) 1 [[arXiv:0710.1818](#)].
- [60] M. Gualtieri, *Generalized complex geometry*, [[math.DG/0401221](#)].
- [61] M. Graña, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [[hep-th/0509003](#)].
- [62] A. Collinucci and J. Evslin, *Twisted homology*, *JHEP* **03** (2007) 058 [[hep-th/0611218](#)].
- [63] P.B. Garrett, *Buildings and classical groups*, Chapman & Hall/CRC (1997).
- [64] K.S. Brown, *Buildings*, Springer (1989).
- [65] A.J. Feingold and H. Nicolai, *Subalgebras of hyperbolic Kac-Moody algebras*, [math.0303179/](#).
- [66] M.C.N. Cheng and E.P. Verlinde, *Wall crossing, discrete attractor flow and borchers algebra*, [arXiv:0806.2337](#).
- [67] M. Henneaux, M. Leston, D. Persson and P. Spindel, *Geometric configurations, regular subalgebras of E_{10} and M-theory cosmology*, *JHEP* **10** (2006) 021 [[hep-th/0606123](#)].
- [68] V.A. Belinskii and I.M. Khalatnikov, *Effect of scalar and vector fields on the nature of the cosmological singularity*, *Sov. Phys. JETP* **36** (1973) 591.
- [69] J. Brown, S. Ganguli, O.J. Ganor and C. Helfgott, *E_{10} orbifolds*, *JHEP* **06** (2005) 057 [[hep-th/0409037](#)].
- [70] M. Bagnoud and L. Carlevaro, *Hidden Borchers symmetries in \mathbb{Z}_n orbifolds of M-theory and magnetized D-branes in type O' orientifolds*, *JHEP* **11** (2006) 003 [[hep-th/0607136](#)].
- [71] S. Mizoguchi, K. Mohri and Y. Yamada, *Five-dimensional supergravity and hyperbolic Kac-Moody algebra G_2^H* , *Class. and Quant. Grav.* **23** (2006) 3181 [[hep-th/0512092](#)].
- [72] M.R. Gaberdiel, D.I. Olive and P.C. West, *A class of Lorentzian Kac-Moody algebras*, *Nucl. Phys.* **B 645** (2002) 403 [[hep-th/0205068](#)].
- [73] S. de Buyl, *Kac-Moody algebras in M-theory*, Ph.D. Thesis, Université Libre de Bruxelles (2006) [[hep-th/0608161](#)].